trans.firsto First–order systems in transient response

1 First order systems have input-output differential equations of the form

$$\tau \frac{dy}{dt} + y = b_1 \frac{du}{dt} + b_0 u \tag{1}$$

with $\tau \in \mathbb{R}$ called the time constant of the system. Systems with a single energy storage element—such as those with electrical or thermal capacitance—can be modeled as first-order.

time constant

2 The characteristic equation yields a single root $\lambda = -1/\tau$, so the homogeneous solution y_h, for constant $\kappa \in \mathbb{R}$, is

homogeneous solution

$$\boxed{y_h = k e^{-t/\tau}}$$

Free response

3 The free response y_{fr} of a system is its response to initial conditions and no forcing ($f(t) = 0$). This is useful for two reasons:

free response y_{fr}

1. perturbations of the system from equilibrium result in free response and
2. from superposition, the free response can be added to a forced response to find the specific response: $y(t) = y_{fr}(t) + y_{fo}(t)$. This allows us to use tables of solutions like Table firsto.1 to construct solutions for systems with nonzero initial conditions with forcing.

4 The free response is found by applying initial conditions to the homogeneous solution. With initial condition $y(0)$, the free response is

$$y_{fr}(t) = y(0)\, e^{-t/\tau}, \tag{2}$$

which begins at $y(0)$ and decays exponentially to zero.

Step response

5 In what follows, we develop forced response y_{fo} solutions, which are the specific solution responses of systems to given inputs and zero initial conditions: all initial conditions set to zero.

forced response y_{fo}

zero initial conditions

6 If we consider the common situation that $b_1 = 0$ and $u(t) = K u_s(t)$ for some $K \in \mathbb{R}$, the solution to Equation 1 is

$$\boxed{y_{fo}(t) = K b_0 (1 - e^{-t/\tau})}$$

if $K = 1$

$y_{fo}(t) = b_0(1 - e^{-t/\tau})$

$y_1 = 1 - e^{-t/\tau}$ $\qquad \dot{y}_1 = \frac{1}{\tau} e^{-t/\tau}$

$y_2 = b_1 \dot{y}_1 + b_0 y_1$

$= b_1 \frac{1}{\tau} e^{-t/\tau} + b_0(1 - e^{-t/\tau}) = b_1 \frac{1}{\tau} e^{-t/\tau} + b_0 - b_0 e^{-t/\tau}$

$= (\frac{b_1}{\tau} - b_0) e^{-t/\tau} + b_0$

$= b_0 - (b_0 - \frac{b_1}{\tau}) e^{-t/\tau}$

The non-steady term is simply a constant scaling of a decaying exponential.

7 A plot of the step response is shown in Figure firsto.1. As with the free response, within 5τ the transient response is less than 1% of the difference between $y(0)$ and steady-state.

Impulse and ramp responses

8 The response to all three singularity inputs are included in Table firsto.1. These can be combined with the free response of Equation 2 using superposition. Results could be described as bitchin'.

bitchin'

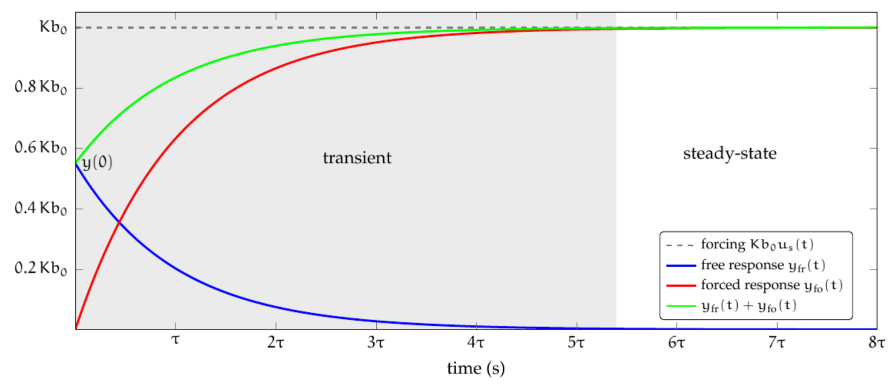

Figure firsto.1: free and forced responses and their sum for a first order system with input $u(t) = K u_s(t)$, initial condition $y(0)$, and $b_1 = 0$.

Table firsto.1: first-order system characteristic and total forced responses for singularity inputs. The relevant differential equation is of the standard form $\tau\dot{y} + y = f$.

$u(t)$	characteristic response $f(t) = u(t)$	total forced response y_{fo} for $t \geqslant 0$ $f(t) = b_1\dot{u} + b_0 u$
$\delta(t)$	$\frac{1}{\tau} e^{-t/\tau}$	$\frac{b_1}{\tau}\delta(t) + \left(\frac{b_0}{\tau} - \frac{b_1}{\tau^2}\right) e^{-t/\tau}$
$u_s(t)$	$1 - e^{-t/\tau}$	$b_0 - \left(b_0 - \frac{b_1}{\tau}\right) e^{-t/\tau}$
$u_r(t)$	$t - \tau(1 - e^{-t/\tau})$	$b_0 t + (b_1 - b_0\tau)(1 - e^{-t/\tau})$

Example trans.firsto–1 **re: RC–circuit response the easy way**

Consider a parallel RC-circuit with input current $I_S(t) = 2u_s(t)$ A, initial capacitor voltage $v_C(0) = 3$ V, resistance $R = 1000\ \Omega$, and capacitance $C = 1$ mF. Proceeding with the usual analysis would produce the io differential equation

$$C\frac{dv_C}{dt} + v_C/R = I_S.$$

Use Table firsto.1 to find $v_C(t)$.

$u = I_s = 2u_s$

$y = V_c$

$C\frac{dv_c}{dt} + \frac{V_c}{R} = I_s$

$RC\frac{dv_c}{dt} + V_c = RI_s$

$\tau = RC$ $\qquad f(t) = 2R\,u_s(t)$

$y_{fr} = y(0) e^{-t/\tau}$ $\qquad y(0) = V_c(0) = 3$

$= 3e^{-t/\tau}$

$y_{ch} = 1 - e^{-t/\tau}$

$y_{fo} = 2R(1 - e^{-t/\tau})$

$y(t) = y_{fr}(t) + y_{fo}(t)$

$= 3e^{-t/\tau} + 2R(1 - e^{-t/\tau})$

$= 2R + (3 - 2R)e^{-t/\tau}$