trans.secondo Second−order systems in transient response

1 Second-order systems have input-output differential equations of the form

$$\frac{d^2y}{dt^2} + 2\zeta\omega_n \frac{dy}{dt} + \omega_n^2 y = f(t) \quad (1)$$

$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0$

where ω_n is called the natural frequency, ζ is called the (dimensionless) damping ratio, and f is a forcing function that depends on the input u as

natural frequency ω_n
damping ratio ζ

$$f(t) = b_2\frac{d^2u}{dt^2} + b_1\frac{du}{dt} + b_0 u. \quad (2)$$

Systems with two energy storage elements—such as those with an inertial element and a spring-like element—can be modeled as second-order.

2 For distinct roots ($\lambda_1 \neq \lambda_2$), the homogeneous solution is, for some real constants κ_1 and κ_2,

$$y_h(t) = \kappa_1 e^{\lambda_1 t} + \kappa_2 e^{\lambda_2 t} \quad (3)$$

where

$$\lambda_1, \lambda_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}. \quad (4)$$

Free response

3 The free response y_{fr} is found by applying initial conditions to the homogeneous solution. With initial conditions $y(0)$ and $\dot{y}(0) = 0$, the free response is

$$y_{fr}(t) = y(0)\frac{1}{\lambda_2 - \lambda_1}\left(\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t}\right). \quad (5)$$

There are five possibilities for the location of the roots λ_1 and λ_2, all determined by the value of ζ.

$\zeta \in (-\infty, 0)$: **unstable** This case is very undesirable because it means our system is unstable and, given any nonzero input or output, will explode to infinity.

explode

$\zeta = 0$: **undamped** An undamped system will oscillate forever if perturbed from zero output.

$\zeta \in (0, 1)$: **underdamped** Roughly speaking, underdamped systems oscillate, but not forever. Let's consider the form of the solution for initial conditions and no forcing. The roots of the characteristic equation are

$$\lambda_1, \lambda_2 = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -\zeta\omega_n \pm j\omega_d \quad (6)$$

where the damped natural frequency ω_d is defined as

damped natural frequency ω_d

$$\omega_d \equiv \omega_n\sqrt{1-\zeta^2}. \quad (7)$$

From Equation (5) for the free response, using Euler's formulas to write in terms of trigonometric functions, and the initial conditions $y(0)$ and $\dot{y}(0) = 0$, we have

$$y_{fr}(t) = y(0)\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\cos(\omega_d t + \psi) \quad (8)$$

where the phase ψ is

$$\psi = -\arctan\frac{\zeta}{\sqrt{1-\zeta^2}}. \quad (9)$$

This is an oscillation that decays to the value it oscillates about, $y(t)|_{t\to\infty} = 0$. So any perturbation of an underdamped system will result in a decaying oscillation about equilibrium.

$\zeta = 1$: **critically damped** In this case, the roots of the characteristic equation are equal:

$$\lambda_1 = \lambda_2 = -\omega_n \quad (10)$$

So we must modify Equation 3 with a factor of t for the homogeneous solution. The free response for initial conditions $y(0)$ and $\dot{y}(0) = 0$, we have

$$y_{fr}(t) = y(0)(1 + \omega_n t)e^{-\omega_n t}. \quad (11)$$

This decays without oscillation, but just barely.

$\zeta \in (1, \infty)$: **overdamped** Here the roots of the characteristic equation are distinct and real. From Equation (5) with free response to initial conditions $y(0)$ and $\dot{y}(0) = 0$, we have the sum of two decaying real exponentials. This response will neither overshoot nor oscillate—like the critically damped case—but with even less gusto.

4 Figure secondo.1 displays the free response results. Note that a small damping ratio results in overshooting and oscillation about the equilibrium value. In contrast, large damping ratio results in neither overshoot nor oscillation. However, both small and large damping ratios yield responses that take longer durations to approach equilibrium than damping ratios near unity. In terms of system performances, there are tradeoffs on either side of $\zeta = 1$. Slightly less than one yields faster responses that overshoot a small amount. Slightly greater than one yields slower responses less prone to oscillation.

Step response

5 Second-order systems are subjected to a variety of forcing functions f. In this lecture, we examine a common one: step forcing. In what follows, we develop forced response y_{fo} solutions.

forced response y_{fo}

6 Unit step forcing of the form $f(t) = u_s(t)$, where u_s is the unit step function, models abrupt changes to the input. The solution is found by applying zero initial conditions ($y(0) = 0$ and $\dot{y}(0) = 0$) to the general solution. If the roots of the characteristic equation λ_1 and λ_2 are distinct, the forced response is

$$y_{fo}(t) = \frac{1}{\omega_n^2}\left(1 - \frac{1}{\lambda_2 - \lambda_1}\left(\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t}\right)\right) \quad (12)$$

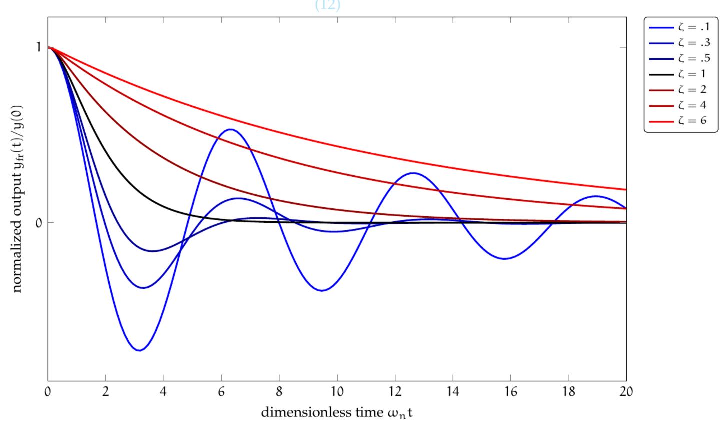

Figure secondo.1: free response $y_{fr}(t)$ of a second-order system with initial conditions $y(0)$ and $\dot{y}(0) = 0$ for different values of ζ. Underdamped, critically damped, and overdamped cases are displayed.

where

$$\lambda_1, \lambda_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}. \quad (13)$$

Once again, there are five possibilities for the location of the roots of the characteristic equation λ_1 and λ_2, all determined by the value of ζ. However, there are three stable cases: underdamped, critically damped, and overdamped.

$\zeta \in (0, 1)$ **underdamped** In this case, the roots are distinct and complex:

$$\lambda_1, \lambda_2 = -\zeta\omega_n \pm j\omega_d. \quad (14)$$

From Equation 12, the forced step response is

$$y_{fo}(t) = \frac{1}{\omega_n^2}\left(1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\cos(\omega_d t + \psi)\right) \quad (15)$$

where the phase ψ is

$$\psi = -\arctan\frac{\zeta}{\sqrt{1-\zeta^2}}. \quad (16)$$

This response overshoots, oscillates about, and decays to $1/\omega_n^2$.

$\zeta = 1$ **critically damped** The roots are equal and real:

$$\lambda_1, \lambda_2 = -\omega_n \quad (17)$$

so the forced step of Equation 12 must be modified; it reduces to

$$y_{fo}(t) = \frac{1}{\omega_n^2}\left(1 - e^{-\omega_n t}(1 + \omega_n t)\right). \quad (18)$$

This response neither oscillates nor overshoots its steady-state of $\frac{1}{\omega_n^2}$, but just barely.

$\zeta \in (1, \infty)$ **overdamped** In this case, the roots are distinct and real, given by Equation 13. The forced step given by Equation 12 is the sum of two decaying real exponentials. These responses neither overshoot nor oscillate about their steady-state of $1/\omega_n^2$. With increasing ζ, approach to steady-state slows.

7 Figure secondo.2 displays the forced step response results. These responses are inverted versions of the free responses of Lecture trans.secondo. Note that a small damping ratio results in overshooting and oscillation about the steady-state value. In contrast, large damping ratio results in neither overshoot nor oscillation. However, both small and large damping ratios yield responses that take longer durations to approach equilibrium than damping ratios near unity. For this reason, the damping ratio of a system should be close to $\zeta = 1$. There are tradeoffs on either side of one. Slightly less yields faster responses that overshoot a small amount. Slightly greater than one yields slower responses less prone to oscillation.

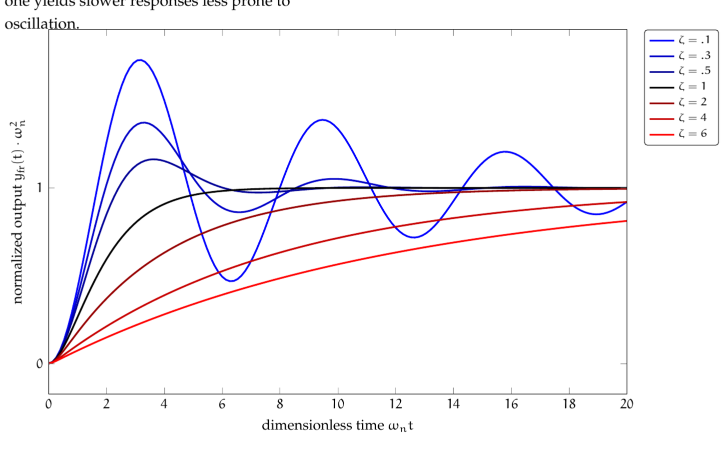

Figure secondo.2: forced step response $y_{fo}(t)$ of a second-order system for different values of ζ. Underdamped, critically damped, and overdamped cases are displayed.

Impulse and ramp responses

8 The response to all three singularity inputs are included in Table secondo.1. These can be combined with the free response of Equation 2 using superposition.

An example with superposition

9 The results of the above are powerful not so much in themselves, but when they are wielded with the principle of superposition, as the following example shows.

Example trans.secondo−1 **re: MRFM cantilever beam with initial condition and forcing**

In magnetic resonance force microscopy (MRFM), the primary detector is a small cantilever beam with a magnetic tip. Model the beam as a spring-mass-damper system with mass $m = 6$ pg,[a] spring constant $k = 15$ mN/m,

Table secondo.1: responses of system $\frac{d^2y}{dt^2} + 2\zeta\omega_n\frac{dy}{dt} + \omega_n^2 y = f$ to different singularity forcing. Note that $\tau_1 = -1/\lambda_1$, $\tau_2 = 1/\lambda_2$, and $\psi = -\arctan(\zeta/\sqrt{1-\zeta^2})$.

damping	f(t)	characteristic response
$0 \leqslant \zeta < 1$	$\delta(t)$	$\frac{e^{-\zeta\omega_n t}}{\omega_n\sqrt{1-\zeta^2}}\sin(\omega_d t)$
	$u_s(t)$	$\frac{1}{\omega_n^2}\left(1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\cos(\omega_d t + \psi)\right)$
	$u_r(t)$	$\frac{1}{\omega_n^2}\left(t + \frac{e^{-\zeta\omega_n t}}{\omega_n}\left(2\zeta\cos\omega_d t + \frac{2\zeta^2-1}{\sqrt{1-\zeta^2}}\sin\omega_d t\right) - \frac{2\zeta}{\omega_n}\right)$
$\zeta = 1$	$\delta(t)$	$te^{-\omega_n t}$
	$u_s(t)$	$\frac{1}{\omega_n^2}\left(1 - e^{-\omega_n t} - \omega_n te^{-\omega_n t}\right)$
	$u_r(t)$	$\frac{1}{\omega_n^2}\left(t + \frac{2}{\omega_n}e^{-\omega_n t} + te^{-\omega_n t} - \frac{2}{\omega_n}\right)$
$\zeta > 1$	$\delta(t)$	$\frac{1}{2\omega_n\sqrt{\zeta^2-1}}\left(e^{-t/\tau_1} - e^{-t/\tau_2}\right)$
	$u_s(t)$	$\frac{1}{\omega_n^2}\left(1 - \frac{\omega_n}{2\sqrt{\zeta^2-1}}\left(\tau_1 e^{-t/\tau_1} - \tau_2 e^{-t/\tau_2}\right)\right)$
	$u_r(t)$	$\frac{1}{\omega_n^2}\left(t - \frac{2\zeta}{\omega_n} + \frac{\omega_n}{2\sqrt{\zeta^2-1}}\left(\tau_1^2 e^{-t/\tau_1} - \tau_2^2 e^{-t/\tau_2}\right)\right)$

and damping coefficient $B = 37.7 \cdot 10^{-15}$ N·s/m. Magnetic input forces on the beam $u(t)$ are applied directly to the magnetic tip. That is, a Newtonian force-analysis yields the input-output ODE

$$m\ddot{y} + B\dot{y} + ky = u,$$

where y models the motion of the tip.

1. What is the natural frequency ω_n?
2. What is the damping ratio ζ?
3. Find the free response for initial conditions $y(0) = 10$ nm and $\dot{y}(0) = 0$.
4. Find the impulse (forced) response for input $u(t) = 3\delta(t)$.
5. Find the total response for the initial condition and forcing, from above, are both applied.

a. One pg $= 10^{-12}$g $= 10^{-15}$kg.

$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = f(t)$

$\ddot{y} + \frac{B}{m}\dot{y} + \frac{k}{m}y = \frac{f(t)}{m}$

$6\,pg = 6\times10^{-15}$ kg

$15\,\frac{mN}{m} = 15\times10^{-3}\,\frac{N}{m}$

$\frac{k}{m} = \omega_n^2 \Rightarrow \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{6\times10^{-15}}{15\times10^{-3}}} = 62.8\times10^3\,\frac{rad}{s}$

$\frac{B}{m} = 2\zeta\omega_n \Rightarrow \frac{B}{2m\omega_n} = \frac{37.7\times10^{-15}}{2(6\times10^{-15})(62.8\times10^3)} = 0.00005$

if $\zeta = 0.8$ $\omega_d = \omega_n\sqrt{1-0.8^2} = 0.6\,\omega_n$

$\omega_d = \omega_n\sqrt{1-\zeta^2} = 62.8\times10^3\sqrt{1-(5\times10^{-5})^2} = 62.8\times10^3\,\frac{rad}{s}$

$y_{fr}(t) = y(0)\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\cos(\omega_d t + \psi)$

$\psi = -\mathrm{atan}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right) = -5\times10^{-5}$ rad

$y(0) = 10\,nm = 10\times10^{-9}$ m

$y_{fr}(t) = 10\times10^{-9}\frac{e^{-3.14t}}{\sqrt{1-(5\times10^{-5})^2}}\cos(62.8\times10^3 t - 5\times10^{-5})$

$y_{fo}(t) = \frac{3}{m}\frac{e^{-\zeta\omega_n t}}{\omega_n\sqrt{1-\zeta^2}}\sin(\omega_d t) = 7.96\times10^9 e^{-3.14t}\sin(62.8\times10^3 t)$

$y(t) = \left(10\times10^{-9}\cos(62.8\times10^3 t - 5\times10^{-5}) + 7.96\times10^9\sin(62.8\times10^3 t)\right)e^{-3.14t}$