

ssresp.diag Diagonalizing basis

1 It is useful to transform a system's state vector x into a special basis that diagonalizes—leaves nonzero components along only the diagonal—the system's A -matrix. For systems with n distinct eigenvalues, to which we limit ourselves in this discussion,⁴ this is always possible. In diagonalized form, it will be relatively easy to solve for the state transition matrix Φ .

diagonalizes

4. See Appendix a.b.v. eig for general considerations.

Changing basis in the state equation

2 As with all basis transformations, the basis transformation we seek can be written

$$x = Px' \Rightarrow x' = P^{-1}x, \quad (1)$$

transformation matrix

where P is the transformation matrix, x is a representation of the state vector in the original basis, and x' is a representation of the state vector in the new basis.⁵

5. We are being a bit fast-and-loose with terminology here: a vector is an object that does not change under basis transformation, only its components and basis vectors do. However, we use the common notational and terminological abuses.

3 Substituting this transformation into the standard linear state-model equations yields the model

$$x' = \underbrace{P^{-1}AP}_{A'}x' + \underbrace{P^{-1}B}_{B'}u \quad (2a)$$

$$y = \underbrace{CP}_{C'}x' + \underbrace{D}_{D'}u. \quad (2b)$$

Model and eigenvalue matrices

4 Let a state equation have matrix A with n distinct eigenvalues $\{\lambda_i\}$ and eigenvectors $\{m_i\}$. Let the eigenvalue matrix Λ be defined as

eigenvalue matrix

$$\Lambda = \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & \lambda_n \end{bmatrix}$$

5 Furthermore, let the modal matrix M be defined as

modal matrix

$$M = \begin{bmatrix} | & | & & | \\ m_1 & m_2 & \dots & m_n \\ | & | & & | \end{bmatrix} \quad (3)$$

Diagonalization of the state equation

6 Let the modal matrix M be the transformation matrix for our state-model. Then $x' = M^{-1}x$.

6. As long as there are n distinct eigenvalues, M is invertible.

7 The state equation becomes

$$x' = M^{-1}AMx' + M^{-1}Bu. \quad (4)$$

The eigenproblem implies that

$$\begin{aligned} A[m_1 \ m_2 \ \dots \ m_n] &= [m_1 \ m_2 \ \dots \ m_n]\Lambda \\ AM &= M\Lambda \\ M^{-1}AM &= M^{-1}M\Lambda \\ &= \Lambda \end{aligned}$$

That is, $A' = \Lambda'$. Recall that Λ is diagonal; therefore, we have diagonalized the state-space model. In full-form, the diagonalized model is

diagonalized

$$x' = \underbrace{\Lambda}_{A'}x' + \underbrace{M^{-1}B}_{B'}u \quad (5a)$$

$$y = \underbrace{CM}_{C'}x' + \underbrace{D}_{D'}u. \quad (5b)$$

Computing the state transition matrix

8 Recall our definition of the state transition matrix $\Phi(t) = e^{At}$. Directly applying this to the diagonalized system of Eq. 5,

$$\begin{aligned} \Phi'(t) &= e^{\Lambda t} \quad (6a) \\ &= \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} x'(0) \\ x'(0) \\ \vdots \\ x'(0) \end{bmatrix} \end{aligned}$$

diagonal property

In this last equality, we have used the diagonal property of the state transition matrix, defined in Lec. ssresp.response.

9 Recall that the free response solution to the state equation is given by the initial condition and state transition matrix, so

$$x'_f(t) = \Phi'(t)x'(0) \quad (7a)$$

$$= x'_1(0)e^{\lambda_1 t} + x'_2(0)e^{\lambda_2 t} + \dots + x'_n(0)e^{\lambda_n t} \quad (7b)$$

where the initial conditions are $x'(0) = M^{-1}x(0)$. We have completely decoupled each state's free response, one of the remarkable qualities of the diagonalized system.

10 At this point, one could simply solve the diagonalized system for $x'(t)$, then convert the solution to the original basis with $x(t) = Mx'(t)$.

11 Sometimes, we might prefer to transform the diagonalized-basis state transition matrix into the original basis. The following is a derivation of that transformation.

12 Beginning with the free response solution in the diagonalized-basis and transforming the equation into the original basis, we find an expression for the original state transition matrix, as follows.

$$\begin{aligned} x'_f(t) &= \underline{\mathcal{F}}'(t)x'(0) \\ M^{-1}x_f(t) &= \underline{\mathcal{F}}'(t)M^{-1}x(0) \\ x_f(t) &= M\underline{\mathcal{F}}'(t)M^{-1}x(0) \end{aligned}$$

This last expression is just the free response solution in the original basis, so we can identify

$$\Phi(t) = M\underline{\mathcal{F}}'(t)M^{-1}. \quad (8)$$

This is a powerful result. Eq. 8 is the preferred method of deriving the state transition matrix for a given system. The eigenvalues give Φ' and the eigenvectors give M .

Example ssresp.diag-1

re: state free response

For the state equation

$$\dot{x} = \begin{bmatrix} -2 & 2 \\ 2 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u$$

find the state's free response to initial condition $x(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

$$A = \begin{bmatrix} -2 & 2 \\ 2 & -3 \end{bmatrix} \quad \lambda = -4, 56, -0.79$$

$$\underline{\mathcal{F}}'(t) = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} M = \begin{bmatrix} -0.62 & -0.79 \\ 0.79 & -0.62 \end{bmatrix} \quad M^{-1} = \begin{bmatrix} -0.62 & 0.79 \\ -0.79 & -0.62 \end{bmatrix}$$

$$\underline{\mathcal{F}}(t) = M\underline{\mathcal{F}}'(t)M^{-1} = \begin{bmatrix} -0.62 & -0.79 \\ 0.79 & -0.62 \end{bmatrix} \begin{bmatrix} e^{-4.56t} & 0 \\ 0 & e^{-0.79t} \end{bmatrix} \begin{bmatrix} -0.62 & 0.79 \\ -0.79 & -0.62 \end{bmatrix}$$

$$x_f(t) = \underline{\mathcal{F}}(t)x(0) = \begin{bmatrix} -0.62 & -0.79 \\ 0.79 & -0.62 \end{bmatrix} \begin{bmatrix} e^{-4.56t} & 0 \\ 0 & e^{-0.79t} \end{bmatrix} \begin{bmatrix} -0.62 & 0.79 \\ -0.79 & -0.62 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$