ssresp.eig Linear algebraic eigenproblem

1 The linear algebraic eigenproblem can be simply stated. For $n \times n$ real matrix A, $n \times 1$ complex vector m, and $\lambda \in \mathbb{C}$, m is defined as an eigenvector of A if and only if it is nonzero and

eigenproblem

eigenvector

$$Am = \lambda m \tag{1}$$

for some λ, which is called the corresponding eigenvalue. That is, m is an eigenvector of A if its linear transformation by A is equivalent to its scaling; i.e. an eigenvector of A is a vector of which A changes the length, but not the direction.

eigenvalue

2 Since a matrix can have several eigenvectors and corresponding eigenvalues, we typically index them with a subscript; e.g. m_i pairs with λ_i.

Solving for eigenvalues

Eq. 1 can be rearranged:

$$(\lambda I - A)m = 0. \tag{2}$$

For a nontrivial solution for m,

$$\det(\lambda I - A) = 0, \tag{3}$$

which has as its left-hand-side a polynomial in λ and is called the characteristic equation. We define eigenvalues to be the roots of the characteristic equation.

characteristic equation

eigenvalues

Box ssresp.1 eigenvalues and roots of the characteristic equation

If A is taken to be the linear state-space representation A, and the state-space model is converted to an input-output differential equation, the resulting ODE's "characteristic equation" would be identical to this matrix characteristic equation. Therefore, everything we already understand about the roots of the "characteristic equation" of an i/o ODE—especially that they govern the transient response and stability of a system—holds for a system's A-matrix eigenvalues.

3 Here we consider only the case of n distinct eigenvalues. For eigenvalues of (algebraic) multiplicity greater than one (i.e. repeated roots), see the discussion of Appendix adv.eig.

Solving for eigenvectors

4 Each eigenvalue λ_i has a corresponding eigenvector m_i. Substituting each λ_i into Eq. 2, one can solve for a corresponding eigenvector. It's important to note that an eigenvector is unique within a scaling factor. That is, if m_i is an eigenvector corresponding to λ_i, so is $3m_i$.[3]

3. Also of note is that λ_i and m_i can be complex.

Example ssresp.eig-1 re: eigenproblem for a 2×2 matrix

Let

$$A = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}.$$

Find the eigenvalues and eigenvectors of A.

$$\det(\lambda I - A) = 0$$

$$\det\left(\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} \lambda-2 & 4 \\ 1 & \lambda+1 \end{bmatrix}\right) = 0$$

$$(\lambda-2)(\lambda+1) - 4 = 0$$

$$\lambda^2 - 2\lambda + \lambda - 2 - 4 = 0$$

$$\lambda^2 - \lambda - 6 = 0$$

$$(\lambda - 3)(\lambda + 2) = 0$$

$$\lambda = 3, -2$$

$$(\lambda I - A)m = 0$$

$\lambda_1 = 3$

$$\left(\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}\right)\begin{bmatrix} m_{11} \\ m_{21} \end{bmatrix} = \vec{0}$$

$$\begin{bmatrix} 1 & 4 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} m_{11} \\ m_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$m_{11} + 4m_{12} = 0$ $\quad m_{11} = -4m_{12}$

$m_{11} + 4m_{12} = 0$

$$m_1 = a\begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

$\lambda_2 = -2$

$$\left(\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} - \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}\right)\begin{bmatrix} m_{12} \\ m_{22} \end{bmatrix} = \vec{0}$$

$$\begin{bmatrix} -4 & 4 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} m_{12} \\ m_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-4m_{12} + 4m_{22} = 0$ $\quad m_{12} = m_{22}$

$1m_{12} - 1m_{22} = 0$ $\quad m_{12} = m_{22}$

$$m_2 = b\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

5 Several computational software packages can easily solve for eigenvalues and eigenvectors. See Lec. ssresp.eigcomp for instruction for doing so in Matlab and Python.