freq.sin Sinusoidal input, frequency response

1 In this lecture, we explore the relationship—which turns out to be pretty chummy—between a system's frequency response function $H(j\omega)$ and its sinusoidal forced response.

2 Let's build from the frequency response function $H(j\omega)$ definition:

$$\begin{aligned} y(t) &= \mathcal{F}^{-1}Y(\omega) && \text{(1a)} \\ &= \mathcal{F}^{-1}(H(j\omega)U(\omega)). && \text{(1b)} \end{aligned}$$

We take the input to be sinusoidal, with amplitude $A \in \mathbb{R}$, angular frequency ω_0, and phase ψ:

$$u(t) = A\cos(\omega_0 t + \psi). \tag{2}$$

The Fourier transform of the input, $U(\omega)$, can be constructed via transform identities from Table ft.1. This takes a little finagling. Let

$$\begin{aligned} p(t) &= Aq(t), && \text{(3a)} \\ q(t) &= r(t - t_0), \text{ and} && \text{(3b)} \\ r(t) &= \cos\omega_0 t, \text{ where} && \text{(3c)} \\ t_0 &= -\psi/\omega_0. && \text{(3d)} \end{aligned}$$

The corresponding Fourier transforms, from Table ft.1, are

$$\begin{aligned} P(\omega) &= AQ(\omega), && \text{(4a)} \\ Q(\omega) &= e^{-j\omega t_0}R(\omega), \text{ and} && \text{(4b)} \\ R(\omega) &= \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0). && \text{(4c)} \end{aligned}$$

Putting these together,

$$\begin{aligned} U(\omega) &= A\pi\left(e^{j\psi\omega/\omega_0}\delta(\omega-\omega_0) + e^{j\psi\omega/\omega_0}\delta(\omega+\omega_0)\right) \\ &= A\pi\left(e^{j\psi}\delta(\omega-\omega_0) + e^{-j\psi}\delta(\omega+\omega_0)\right) \end{aligned}$$

(because δs)

3 And now we are ready to substitute into Eq. 1b; also applying the "linearity" property of the Fourier transform:

$$y(t) = A\pi\left(e^{j\psi}\mathcal{F}^{-1}(H(j\omega)\delta(\omega-\omega_0)) + e^{-j\psi}\mathcal{F}^{-1}(H(j\omega)\delta(\omega+\omega_0))\right). \tag{5}$$

The definition of the inverse Fourier transform gives

$$y(t) = \frac{A}{2}\left(e^{j\psi}\int_{-\infty}^{\infty} e^{j\omega t}H(j\omega)\delta(\omega-\omega_0)d\omega + \right. \\ \left. + e^{-j\psi}\int_{-\infty}^{\infty} e^{j\omega t}H(j\omega)\delta(\omega+\omega_0)d\omega\right). \tag{6}$$

Recognizing that δ is an even distribution ($\delta(t) = \delta(-t)$) and applying the sifting property of δ allows us to evaluate each integral:

$$y(t) = \frac{A}{2}\left(e^{j\psi}e^{j\omega_0 t}H(j\omega_0) + e^{-j\psi}e^{-j\omega_0 t}H(-j\omega_0)\right). \tag{7}$$

Writing H in polar form,

$$y(t) = \frac{A}{2}\left(e^{j(\omega_0 t + \psi)}|H(j\omega_0)|e^{j\angle H(j\omega_0)} + e^{-j(\omega_0 t + \psi)}|H(-j\omega_0)|e^{j\angle H(-j\omega_0)}\right) \tag{8}$$

The Fourier transform is conjugate symmetric—that is, $F(-\omega) = F^*(\omega)$—which allows us to further simply:

$$\begin{aligned} y(t) &= \frac{A|H(j\omega_0)|}{2}\left(e^{j(\omega_0 t+\psi)}e^{j\angle H(j\omega_0)} + e^{-j(\omega_0 t+\psi)}e^{-j\angle H(j\omega_0)}\right) && \text{(9a)} \\ &= A|H(j\omega_0)|\frac{e^{j(\omega_0 t+\psi+\angle H(j\omega_0))} + e^{-j(\omega_0 t+\psi+\angle H(j\omega_0))}}{2}. && \text{(9b)} \end{aligned}$$

Finally, Euler's formula yields something that deserves a box.

Equation 10 sinusoidal response in terms of $H(j\omega)$

For input $A\cos(\omega_0 t + \psi)$ to system $H(j\omega)$, the forced response is

$$y(t) = A|H(j\omega_0)|\cos(\omega_0 t + \psi + \angle H(j\omega_0)).$$

4 This is a remarkable result. For an input sinusoid, a linear system has a forced response that

- is also a sinusoid,
- is at the same frequency as the input,
- differs only in amplitude and phase,
- differs in amplitude by a factor of $|H(j\omega)|$, and
- differs in phase by a shift of $\angle H(j\omega)$.

Now we see one of the key facets of the frequency response function: it governs how a sinusoid transforms through a system. And just think how powerful it will be once we combine it with the powerful principle of superposition and the mighty Fourier series representation of a function—as a "superposition" of sinusoids!