RW

14.1. The phase and amplitude relationships between two sinusoidal variables in a dynamic system are important in fields ranging from sound and video signal processing to elements of automated machinery. Many simple dynamic systems with a single energy storage element have a transfer function:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{K}{\tau s + 1}$$

Consider a system with $K = 2$ and $\tau = 0.1$ s and with an input $u(t) = 5\sin(\omega t)$.

(a) Derive the frequency response function $H(j\omega)$, and express it in terms of the magnitude and phase functions.

(b) Determine the steady-state sinusoidal response for input frequencies

- **i.** $\omega = 5$ rad/s
- **ii.** $\omega = 10$ rad/s
- **iii.** $\omega = 40$ rad/s

For each case sketch both $u(t)$ and $y(t)$ on the same graph.

(c) Comment on the influence of increasing frequency on the amplitude and phase of $y(t)$ with respect to $u(t)$.

a) $H(s) = \frac{K}{\tau s + 1}$ $\quad\quad s \to j\omega$

$$H(j\omega) = \frac{K}{\tau j\omega + 1} \cdot \frac{(1 - \tau j\omega)}{(1 - \tau j\omega)}$$

$$= \frac{K - K\tau\omega j}{1 + (\tau\omega)^2}$$

$$Re(H(j\omega)) = \frac{K}{1 + (\tau\omega)^2} \quad\quad Im(H(j\omega)) = \frac{-K\tau\omega}{1 + (\tau\omega)^2}$$

$$|H(j\omega)| = \sqrt{Re(H(j\omega))^2 + Im(H(j\omega))^2}$$

$$= \frac{\sqrt{K^2 + (K\tau\omega)^2}}{1 + (\tau\omega)^2} = \frac{K\sqrt{1 + (\tau\omega)^2}}{1 + (\tau\omega)^2}$$

$$= \boxed{\frac{K}{\sqrt{1 + (\tau\omega)^2}}} = \frac{2}{\sqrt{1 + 0.01\omega^2}}$$

$$\angle H(j\omega) = \tan^{-1}\left(\frac{Im(H(j\omega))}{Re(H(j\omega))}\right)$$

$$= \tan^{-1}\left(\frac{-K\tau\omega}{K}\right) = \boxed{\tan^{-1}(-\tau\omega)} = \tan^{-1}(-0.1\omega)$$

$$y(t) = 5\,|H(j\omega)|\sin(\omega t + \angle H(j\omega))$$

$\omega = 5$ $\quad\quad y(t) = 5(1.79)\sin(5t - 0.46)$

$\omega = 10$ $\quad\quad y(t) = 5(1.41)\sin(5t - 0.78)$

$\omega = 40$ $\quad\quad y(t) = 5(0.49)\sin(5t - 1.3)$