lap.def Laplace transform and its inverse

The Laplace transform

1 The two-sided definition of the Laplace transform was encountered in Lec. lap.in. This is rarely used in engineering analysis, which prefers the following one-sided transform.[2]

Definition lap.2: Laplace transform

Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a function of time t for which $f(t) = 0$ for $t < 0$. The Laplace transform[3] $\mathcal{L} : T \to S$ of f is defined as[4]

$$(\mathcal{L}f)(s) = \int_0^\infty f(t)e^{-st}\ dt.$$

2 As with the Fourier transform image, it is customary to capitolize the Laplace transform image; e.g.[5]

$$F(s) = (\mathcal{L}f)(s).$$

3 As with the two-sided Laplace transform, if the transform exists, it will do so for some region of convergence (ROC), a subset of the s-plane. It is best practice to report a Laplace transform image paired with its ROC.

4 On the imaginary axis ($\sigma = 0$), $s = j\omega$ and the Laplace transform is

$$(\mathcal{L}f)(s) = \int_0^\infty f(t)e^{-j\omega t}\ dt, \tag{1}$$

which is the one-sided Fourier transform! Therefore, when the Laplace transform exists for a region of convergence that includes the imaginary axis, the one-sided Fourier transform also exists and[6]

$$(\mathcal{F}f)(\omega) = (\mathcal{L}f)(s)|_{s\mapsto j\omega} \tag{2}$$

or, haphazardly using F to denote both transforms,

$$F(\omega) = F(s)|_{s\mapsto j\omega}. \tag{3}$$

Box lap.1 Laplace terminology

The terminology in the literature for the Laplace transform and its inverse, introduced next, is inconsistent. The "Laplace transform" is at once taken to be a function that maps a function of t to a function of s and a particular result of that mapping (technically the image of the map), which is a complex function of s. Even we will say things like "the value of the Laplace transform $F(s)$ at $s = 2 + j4$," by which we really mean the image of the complex function $F(s)|_{s\to 2+j4}$ that was the image of the Laplace transform map of the real function of time $f(t)$. You can see why we shorten it.

The inverse Laplace transform

5 As with the Fourier transform, the Laplace transform has an inverse.

Definition lap.3: Inverse Laplace transform

Let $s \in \mathbb{C}$ be the Laplace s and $F(s)$ a Laplace transform image of real function $f(t)$. The inverse Laplace transform $\mathcal{L}^{-1} : S \to T$ is defined as

$$(\mathcal{L}^{-1}F)(t) = \frac{1}{j2\pi}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}\ ds.$$

As is illustrated in Fig. def.1, it can be shown that the inverse Laplace transform image of a Laplace transform image of $f(t)$ equals $f(t)$ and vice-versa; i.e.

$$(\mathcal{L}^{-1}\mathcal{L}f)(t) = f(t) \quad \text{and}$$
$$(\mathcal{L}\mathcal{L}^{-1}F)(s) = F(s).$$

That is, the inverse Laplace transform is a true inverse. Therefore, we call the Laplace transform and its inverse a pair.

6 A detail view of Fig. def.1 is given in Fig. def.2.

Example lap.def−1

Returning to the troublesome unit step $f(t) = u_s(t)$, calculate its Laplace transform image $F(s)$.

Directly applying the definition,

$$\begin{aligned} F(s) &= \int_0^\infty f(t)e^{-st}\ dt \\ &= \int_0^\infty 1\, e^{-st}\, dt \\ &= \frac{-1}{s}e^{-st}\Big|_{t=0}^{\infty} \\ &= \left(\lim_{t\to\infty} -e^{-(\sigma+j\omega)t}/s\right) - \frac{-1}{s}e^0 && (s = \sigma + j\omega) \\ &= 1/s. && (\sigma > 0) \end{aligned}$$

Note that the limit only converges for $\sigma > 0$, so the region of convergence is the right half s-plane, exclusive of the imaginary axis. This exclusion tells us what we already know, that the Fourier transform u_s does not exist. However, the Laplace transform does exist and is simply $1/s$!

7 Both the Laplace transform and especially its inverse are typically calculated with the help of software and tables such as Table lap.1, which includes specific images and important properties. We will first consider these properties in Lec. lap.pr, then turn to the use of software and tables in Lec. lap.inv where we focus on the more challenging inverse calculation.

2. We will refer exclusively to the one-sided transform as the Laplace transform and will qualify "two-sided" in the other case.

3. Here $T = L^2(0, \infty)$ is the set of square-integrable functions on the positive reals and $S = H^2(\mathbb{C}_+)$ is a Hardy space with square norm on the (complex) right half-plane (Jonathan R. Partington. Linear operators and linear systems: An analytical approach to control theory. London Mathematical Society Student Texts. CUP, 2004. ISBN: 9780521546195, p. 7). This highly mathematical notation highlights the fact that the Laplace transform maps a real function of t to a complex function of s.

4. For more detail, see Rowell and Wormley, (Rowell and Wormley, System Dynamics: An Introduction) Dyke, (P.P.G. Dyke. An Introduction to Laplace Transforms and Fourier Series. 2 edition. Springer Undergraduate Mathematics Series. Springer, 2014. ISBN: 9781447163954) and Mathews and Howell. (John H. Mathews and Russell W. Howell. Complex Analysis for Mathematics and Engineering. 6 edition. Jones and Bartlett Publishers, 2012. ISBN: 9781449604455)

5. Another common notation is $\mathcal{L}(f(t))$.

$s = \sigma + j\omega$

6. The same relation can be shown to hold between the two-sided Fourier and Laplace transforms.

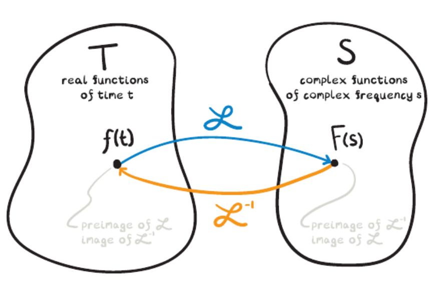

Figure def.1: Laplace transform maps $\mathcal{L}$ and $\mathcal{L}^{-1}$ on the function spaces T and S.

$F(s) = (\mathcal{L}f)(s)$

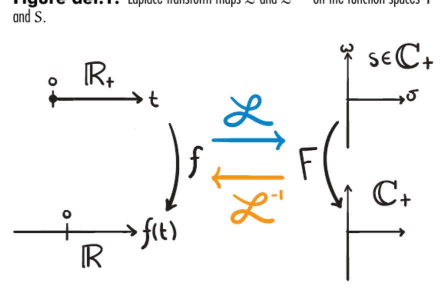

Figure def.2: detail view of Laplace transform maps $\mathcal{L}$ and $\mathcal{L}^{-1}$ along with their image functions f and F.