lap.in Introduction

1 The Laplace transform[1] is a generalized Fourier transform that exists for a much broader class of functions. In fact, every function for which there is a Fourier transform, there is also a Laplace transform—but the reverse does not hold. Its excellence for linear system analysis cannot be overstated, and leads some to undervalue the Fourier transform. However, the Fourier transform is much more conceptually grounded in the frequency domain given that it can be understood as an extension of the Fourier series.

1. See (Rowell and Wormley, System Dynamics: An Introduction, ch. 15) for an introduction that inspires our own.

2 The Laplace transform's conceptual grounding has the same root, but in a less-recognizable form since the explicit frequency variable ω will be consumed by the Laplace transform s, introduced in a moment. But first, we motivate the Laplace transform by identifying a function of great importance to system analysis that does not have a Fourier transform: the unit step function $u_s(t)$.

3 The Fourier transorm of $u_s(t)$ does not exist because its defining improper integral does not converge in the absolute sense—a situation we describe as non-integrable. The Laplace transform does exist for $u_s(t)$ because it patches the Fourier transform integrand with a weighting function $w : \mathbb{R} \to \mathbb{R}$ defined as

non−integrable

$$w(t) = e^{-\sigma t} \quad (1)$$

for $\sigma \in \mathbb{R}$. Clearly such a factor may drive the integrand to zero for some positive σ. Let's take the Fourier transform of a function of time f multiplied by this weightin factor (as a foreshadowing of how the Laplace transform will use it):

$$\begin{aligned} \mathcal{F}(f(t)w(t)) &= \int_{-\infty}^{\infty} f(t)w(t)e^{-j\omega t}\, dt && \text{(FT def.)} \\ &= \int_{-\infty}^{\infty} f(t)e^{-\sigma t}e^{-j\omega t}\, dt && \text{(2a)} \\ &= \int_{-\infty}^{\infty} f(t)e^{-(\sigma+j\omega)t}\, dt. && \text{(2b)} \end{aligned}$$

We see the factor $\sigma + j\omega$ has emerged. This factor also arises in the Laplace transform, so we make an explicit definition.

Definition lap.1: Laplace s

The Laplace $s \in \mathbb{C}$ (a.k.a. complex frequency) is defined as

$$s = \sigma + j\omega$$

for $\sigma, \omega \in \mathbb{R}$.

4 The ubiquity of s has generated a common term called the s-plane, which is used as an alias for the set of complex numbers $\mathbb{C}$, which, when considering its real and imaginary parts to constitute two Cartesian axes (i.e. $\mathbb{R}^2$) charts a plane.

5 Returning to our Fourier transform,

$$\mathcal{F}(f(t)w(t)) = \int_{-\infty}^{\infty} f(t)e^{-st}\, dt. \quad (3)$$

This is sometimes called the two-sided Laplace trnasform, which is rarely used. However, it is instructive to recognize that potentially, for some region of s-values in the complex plane, the transform exists. We call this the region of convergance (ROC) of the transform.

two−sided Laplace trnasform

6 Now consider what happens if $f(t) = u_s(t)$, the unit step that doesn't have a Fourier transform, but the two-sided transform of Eq. 2a yields

$$\mathcal{F}(u_s(t)w(t)) = \int_{-\infty}^{\infty} u_s(t)e^{-\sigma t}e^{-j\omega t}\, dt$$

$$\mathcal{F}(u_s(t)e^{-\sigma t}),$$

a straightup Fourier transform of $u_s(t)e^{-\sigma t}$. Consulting Table ft.1, we see that the transform is

$$\mathcal{F}(u_s(t)e^{-\sigma t}) = \frac{1}{\sigma + j\omega}. = \frac{1}{s} \quad (4)$$

So, although $\mathcal{F}(u_s(t))$ does not exist, $\mathcal{F}(u_s(t)e^{-\sigma t})$ does. This bodes well for the Laplace transform.