# lap.sol Solving io ODEs with Laplace

1 Laplace transforms provide a convenient method for solving input-output (io) ordinary differential equations (ODEs).

2 Consider a dynamic system described by the IO ODE —with $t$ time, $y$ the output, $u$ the input, constant coefficients $a_i, b_j$, order $n$, and $m \leqslant n$ for $n \in \mathbb{N}_0$—as:

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y = b_m\frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1}u}{dt^{m-1}} + \cdots + b_1\frac{du}{dt} + b_0 u. \quad (1)$$

Re-written in summation form,

$$\sum_{i=0}^{n} a_i y^{(i)}(t) = \sum_{j=0}^{m} b_j u^{(j)}(t), \quad (2)$$

where we use Lagrange's notation for derivatives, and where, by convention, $a_n = 1$.

3 The Laplace transform $\mathcal{L}$ of Eq. 2 yields

$$\mathcal{L}\sum_{i=0}^{n} a_i y^{(i)}(t) = \mathcal{L}\sum_{j=0}^{m} b_j u^{(j)}(t) \quad \Rightarrow \quad (3a)$$

$$\sum_{i=0}^{n} a_i \mathcal{L}\left(y^{(i)}(t)\right) = \sum_{j=0}^{m} b_j \mathcal{L}\left(u^{(j)}(t)\right). \quad \text{(linearity)}$$

In the next move, we recursively apply the differentiation property to yield the following

$$\sum_{i=0}^{n} a_i \left( s^i Y(s) + \underbrace{\sum_{k=1}^{i} s^{i-k} y^{\overset{i-k}{\cancel{(k-1)}}}(0)}_{I_i(s)} \right) = \sum_{j=0}^{m} b_j s^j U(s), \quad (4)$$

where terms in $I_i(s)$ arise from the initial conditions. Splitting the left outer sum and solving for $Y(s)$,

$$\sum_{i=0}^{n} a_i s^i Y(s) + \sum_{i=0}^{n} a_i I_i(s) = \sum_{j=0}^{m} b_j s^j U(s) \quad \Rightarrow \quad (5a)$$

$$\sum_{i=0}^{n} a_i s^i Y(s) = \sum_{j=0}^{m} b_j s^j U(s) - \sum_{i=0}^{n} a_i I_i(s) \quad \Rightarrow \quad (5b)$$

$$Y(s)\sum_{i=0}^{n} a_i s^i = U(s)\sum_{j=0}^{m} b_j s^j - \sum_{i=0}^{n} a_i I_i(s) \quad \Rightarrow \quad (5c)$$

$$Y(s) = \underbrace{\frac{\sum_{j=0}^{m} b_j s^j}{\sum_{i=0}^{n} a_i s^i} U(s)}_{Y_{fo}(s)} + \underbrace{\frac{-\sum_{i=0}^{n} a_i I_i(s)}{\sum_{i=0}^{n} a_i s^i}}_{Y_{fr}(s)}. \quad (5d)$$

4 So we have derived the Laplace transform image $Y(s)$ in terms of the forced and free responses (still in the $s$-domain, of course)! For a solution in the time-domain, we must inverse Laplace transform:

**forced**
**free**

$$y(t) = \underbrace{(\mathcal{L}^{-1}Y_{fo})(t)}_{y_{fo}(t)} + \underbrace{(\mathcal{L}^{-1}Y_{fr})(t)}_{y_{fr}(t)}. \quad (6)$$

This is an important result!

### Example lap.sol–1

Consider a system with step input $u(t) = 7u_s(t)$, output $y(t)$, and io ODE

$$\ddot{y} + 2\dot{y} + y = 2u. \quad (7)$$

Solve for the forced response $y_{fo}(t)$ with Laplace transforms.

5 From Eq. 6,

$$\begin{aligned} y_{fo}(t) &= (\mathcal{L}^{-1}Y_{fo})(t) \\ &= \mathcal{L}^{-1}\left(\frac{\sum_{j=0}^{m} b_j s^j}{\sum_{i=0}^{n} a_i s^i} U(s)\right) && \text{(Eq. 5d)} \\ &= \mathcal{L}^{-1}\left(\frac{2}{s^2+2s+1} U(s)\right). && \text{(Eq. 7)} \end{aligned}$$

We can Laplace transform $u(t)$ for $U(s)$:

$$\begin{aligned} U(s) &= (\mathcal{L}u)(s) \\ &= 7(\mathcal{L}u_s)(s) \\ &= \frac{7}{s}, \end{aligned}$$

where the last equality follows from a transform easily found in Table lap.1.

6 Returning to the time response having $U(s)$,

$$\begin{aligned} y_{fo}(t) &= \mathcal{L}^{-1}\left(\frac{2}{s^2+2s+1} U(s)\right) \\ &= \mathcal{L}^{-1}\left(\frac{2}{s^2+2s+1} \cdot \frac{7}{s}\right). \end{aligned}$$

7 We can use Matlab's Symbolic Math toolbox function `partfrac` to perform the partial fraction expansion.

```
syms s 'complex'
Y = 2/(s^2 + 2*s + 1)*7/s;
Y_pf = partfrac(Y)
```

```
Y_pf =

14/s - 14/(s + 1)^2 - 14/(s + 1)
```

Or, a little nicer to look at,

$$Y(s) = 14\left(\frac{1}{s} - \frac{1}{(s+1)^2} - \frac{1}{s+1}\right).$$

Substituting this into our solution,

$$\begin{aligned} y_{fo}(t) &= 14\mathcal{L}^{-1}\left(\frac{1}{s} - \frac{1}{(s+1)^2} - \frac{1}{s+1}\right) && \text{(linearity)} \\ &= 14\left(\mathcal{L}^{-1}\frac{1}{s} - \mathcal{L}^{-1}\frac{1}{(s+1)^2} - \mathcal{L}^{-1}\frac{1}{s+1}\right) \\ &= 14\left(u_s(t) - te^{-t} - e^{-t}\right) && \text{(Table lap.1)} \\ &= 14\left(u_s(t) - (t+1)e^{-t}\right). \end{aligned}$$

So the forced response starts at 0 and decays pseudo-exponentially to a steady 14.

$$\lim_{s\to 0} \cancel{s} \frac{2}{s^2+2s+1}\frac{7}{\cancel{s}}$$

$$\lim_{s\to 0} \frac{2\cdot 7}{s^2+2s+1} = \frac{2\cdot 7}{1} = 14 \checkmark$$