tf.zp Poles and zeros

1 Two important types of objects defined from a transfer function H can be used to characterize a system's behavior: poles and zeros.

poles
zeros

Definition tf.1: poles

Let a system have transfer function H. Its poles are values of s for which

$$|H(s)| \to \infty.$$

$|H(s)| = \left|\frac{f(s)}{0}\right| \to \infty$

2 A transfer function written as a ratio has poles wherever the denominator is zero; that is, s for which[1]

$\boxed{\text{den } H(s) = 0}$

1. It is common to use this as the definition of a pole, which allows us to talk of "pole-zero cancellation." Occasionally we will use this terminology.

Definition tf.2: zeros

Let a system have transfer function H. Its zeros are values of s for which

$$|H(s)| \to 0.$$

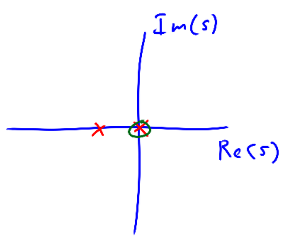
$H(s) = \frac{0}{f(s)} = 0$

3 A transfer function written as a ratio has zeros wherever the numerator is zero; that is, s for which[2]

$\boxed{\text{num } H(s) = 0}$

2. It is common to use this as the definition of a zero, which allows us to talk of "pole-zero cancellation." Occasionally we will use this terminology.

$H(s) = \frac{\cancel{s}}{\cancel{s}(s+1)} = \frac{1}{s+1}$ poles $s = 0, -1$ zeros $s = 0$



4 Given a transfer function H with n poles p_i and ν zeros z_j, we can write, for $K \in \mathbb{R}$,

$$\boxed{H(s) = K \frac{\prod_{j=1}^{\nu} (s - z_j)}{\prod_{i=1}^{n} (s - p_i)}}$$

$H(s) = \frac{1}{s^2 + 2s + 1} = \frac{Y(s)}{U(s)}$

5 Poles and zeros can define a single-input, single-output (SISO) system's dynamic model, within a constant.

6 Recall that, even for multiple-input, multiple-output (MIMO) state-space models, the denominator of every transfer function is the corresponding system's characteristic equation—the roots of which dominate the system's response and are equal to its eigenvalues. It is now time to observe a crucial identity.

$\dot{x} = Ax + Bu$
$y = Cx + Du$ eig(A)

$|\lambda I - A| = 0$

Corollary tf.3: poles = eigenvalues = char. eq. roots

A system's poles equal its eigenvalues equal its characteristic equation roots.

$\mathrm{Re}(\lambda_i) < 0 \quad \forall i \quad$ stable
$\mathrm{Re}(p_i) < 0 \quad \forall i \quad$ stable

7 Therefore, everything we know about a system's eigenvalues and characteristic equation roots is true for a system's poles. This includes that they characterize a system's response (especially its free response) and stability.

Pole-zero plots and stability

8 The complex-valued poles and zeros dominate system behavior via their values and value-relationships. Often, we construct a pole-zero plot—a plot in the complex plane of a system's poles and zeros—such as that of Fig. zp.1.

pole−zero plot

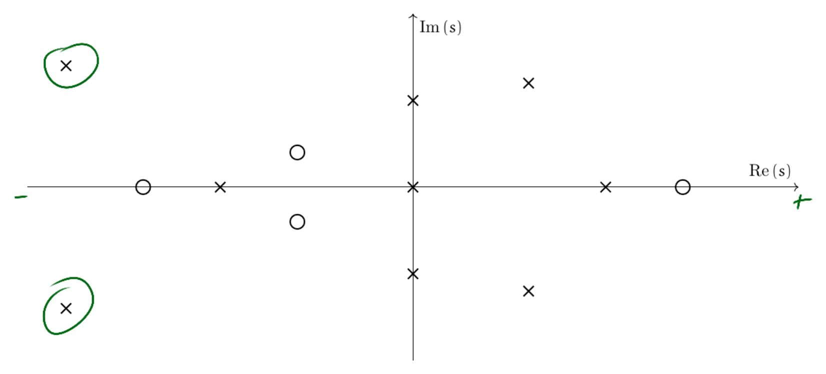


Figure zp.1: a pole-zero plot for a system with nine poles and four zeros. In this example, six of the poles are complex-conjugate pairs and three are real. Three are in the right half-plane, making the system unstable. One zero is in the right half-plane, making the system "minimum phase."

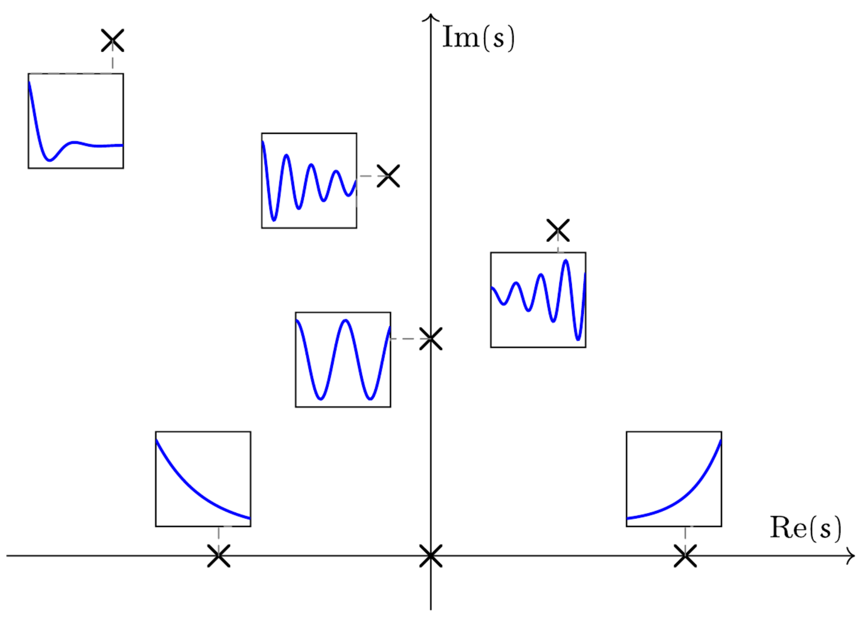


Figure zp.2: free response contributions from poles at different locations. Complex poles contribute oscillating free responses, whereas real poles do not. Left half-plane poles contribute stable responses that decay. Right half-plane poles contribute unstable responses that grow. Imaginary-axis poles contribute marginal stability.

9 From our identification of poles with eigenvalues and roots of the characteristic equation, we can recognize that each pole contributes an exponential response that oscillates if it is complex. There are three stability contribution possibilities for each pole p_i:

- $\mathrm{Re}(p_i) < 0$: a stable, decaying contribution;
- $\mathrm{Re}(p_i) = 0$: a marginally stable, neither decaying nor growing contribution; and
- $\mathrm{Re}(p_i) > 0$: an unstable, growing contribution.

This is explored graphically in Fig. zp.2.

10 Of course, we must not forget that a system's stability is spoiled with a single unstable pole.

11 It can be shown that complex poles and zeros always arise as conjugate pairs. A consequence of this is that the pole-zero plot is always symmetric about the real axis.

real−axis symmetry

Second-order systems

12 Second-order response is characterized by a damping ratio ζ and natural frequency ω_n. These parameters have clear complex-plane "geometric" interpretations, as shown in Fig. zp.3. Pole locations are interpreted geometrically in accordance with their relation to rays of constant damping from the origin and circles of constant natural frequency, centered about the origin.

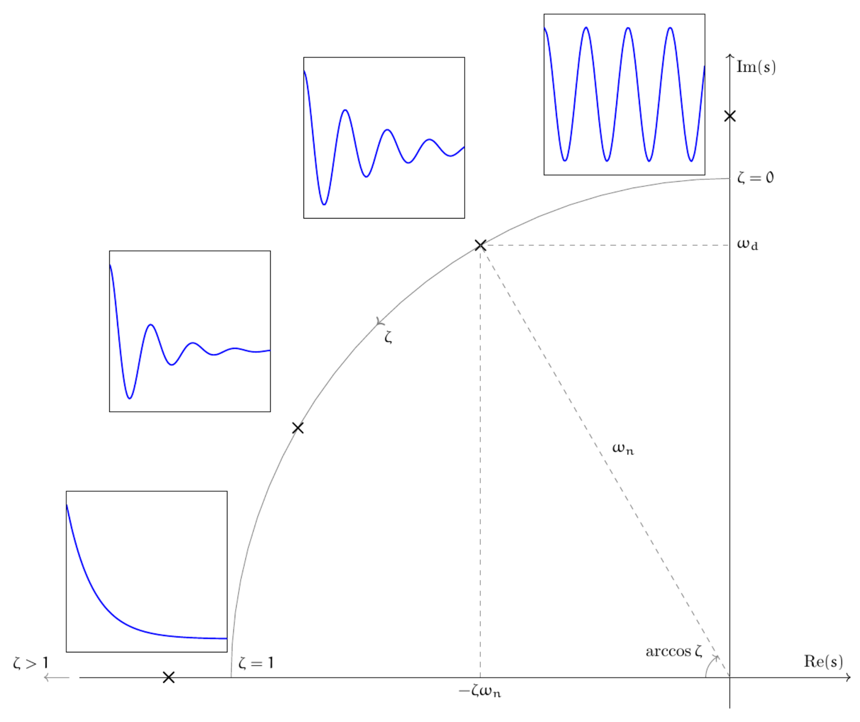


Figure zp.3: second-order free response contributions from poles at different locations, characterized by the damping ratio ζ and natural frequency ω_n. Constant damping occurs along rays from the origin. Constant natural frequency occurs along arcs of constant radius, centered at the origin.