imp.tf Transfer functions via impedance

1 Now the true power of impedance-based modeling is revealed: we can skip a time-domain model (e.g. state-space or io differential equation) and derive a transfer-function model, directly! Before we do, however, let's be sure to recall that a transfer-function model concerns itself with the forced response of a system, ignoring the free response. If we care to consider the free response, we can convert the transfer function model to an io differential equation and solve it.

forced response

2 There are two primary ways impedance-based modeling is used to derive transfer functions. The first and most general is described, here. The second is a shortcut most useful for relatively simple systems; it is described in Lec. imp.divide.

3 In what follows, it is important to recognize that, in the Laplace-domain, every elemental equation is just[1]

$$\mathcal{V} = \mathcal{F}Z, \tag{1}$$

where the across-variable, through-variable, and impedance are all element-specific.

4 This algorithm is very similar to that for state-space models from linear graph models, presented in Lec. ss.nt2ss. In the following, we consider a connected graph with B branches, of which S are sources (split between through-variable sources S_T and across S_A). There are $2B - S$ unknown across- and through-variables, so that's how many equations we need. We have $B - S$ elemental equations and for the rest we will write continuity and compatibility equations. N is the number of nodes.

1. Derive $2B - S$ independent Laplace-domain, algebraic equations from Laplace-domain elemental, continuity, and compatibility equations.
   a) Draw a normal tree. **normal tree**
   b) Write a Laplace-domain elemental equation for each passive element.[2] **elemental equation**
   c) Write a continuity equation for each passive branch by drawing a contour intersecting that and no other branch.[3] **continuity equation**
   d) Write a compatibility equation for each passive link by temporarily "including" it in the tree and finding the compatibility equation for the resulting loop.[4] **compatibility equation**
2. Solve the algebraic system of $2B$ equations and $2B$ unknowns for outputs in terms of inputs, only. Sometimes, solving for all unknowns via the usual methods is easier than trying to cherry-pick the desired outputs.
3. The solution for each output Y_i depends on zero or more inputs U_j. To solve for the transfer function Y_i/U_j, set $U_k = 0$ for all $k \neq j$, then divide both sides of the equation by U_j.

1. In electronics, this is sometimes called "generalized Ohm's law."

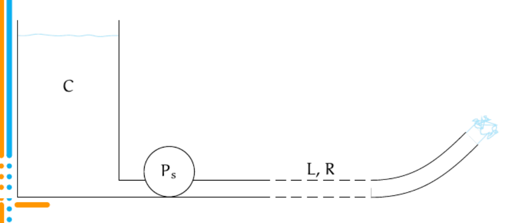
2. There will be $B - S$ elemental equations.

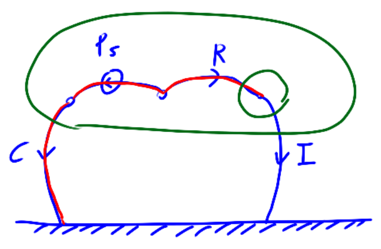
3. There will be $N - 1 - S_A$ independent continuity equations.

4. There will be $B - N + 1 - S_T$ independent compatibility equations.

Example imp.tf–1 re: fire hose

For the schematic of a fire hose connected to a fire truck's reservoir C via pump input P_s, use impedance methods to find the transfer function from P_s to the velocity of the spray. Assume the nozzle's cross-sectional area is A.





$$\text{unknowns} = \begin{bmatrix} p_C \\ p_R \\ p_I \\ q_C \\ q_R \\ q_I \end{bmatrix}$$

$p_C = q_C Z_C$ $\qquad$ $q_R = q_I$

$p_R = q_R Z_R$ $\qquad$ $q_C = q_I$

$p_I = q_I Z_I$ $\qquad$ $p_I = p_C + P_s - p_R$

$$\begin{bmatrix} 1 & 0 & 0 & -Z_C & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ -1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_C \\ p_R \\ p_I \\ q_C \\ q_R \\ q_I \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ P_s \end{bmatrix}$$

$q_I - q_R = 0 \Rightarrow q_I = q_R$

$$q_I = \frac{p_I}{Z_I} = \frac{1}{Z_I}(p_C + P_s - p_R)$$

$$= \frac{1}{Z_I}(q_C Z_C + P_s - q_R Z_R)$$

$$q_I = \frac{1}{Z_I}(q_I Z_C + P_s - q_I Z_R)$$

$$Z_I q_I = q_I Z_C + P_s - q_I Z_R$$

$$Z_I q_I - q_I Z_C + q_I Z_R = P_s$$

$$q_I (Z_I + Z_R - Z_C) = P_s$$

$$q_I = \frac{P_s}{Z_I + Z_R - Z_C}$$

$$\frac{q_I}{P_s} = \frac{1}{Z_I + Z_R + Z_C}$$

$$q_I = \frac{V}{t} = \frac{m^3}{s}$$

$$\frac{V_n}{P_s} = \frac{1}{A}\frac{1}{Z_I + Z_R + Z_C}$$

$$\frac{q_I}{A} = \frac{m^3}{s\,m^2} = \frac{m}{s}$$

$$= \frac{1}{A}\frac{1}{Is + R + 1/Cs}\frac{Cs}{Cs}$$

$$= \frac{1}{A}\frac{Cs}{CIs^2 + RCs + 1}$$