lti.equistab Equilibrium and stability properties

1 For a system with LTI state-space model $\dot{x} = Ax + Bu$, $y = Cx + Du$, the model is in an equilibrium state $\overline{x}$ if $\dot{x} = \vec{0}$. This implies $A\overline{x} + Bu = 0$. For constant input $\overline{u}$, this implies

equilibrium

$$\boxed{A\bar{x} = -Bu}$$

If A is invertible,[1] as is often the case, there is a unique solution for a single equilibrium state:

1. If A is not invertible, the system has at least one eigenvalue equal to zero, which yields an equilibrium subspace equal to an offset (by $B\overline{u}$) of the null space of the state space $\mathbb{R}^n$.

equilibrium

$$\boxed{\bar{x} = -A^{-1}Bu}$$

Definition lti.1: Stability

If x is perturbed from an equilibrium state $\overline{x}$, the response $x(t)$ can:

1. asymptotically return to $\overline{x}$
   asymptotically stable
2. diverge from $\overline{x}$ unstable
3. remain perturbed or oscillate about $\overline{x}$ with a constant amplitude marginally stable

2 A phase portrait is a parametric plot of state variable trajectories, with time implicit. Phase portraits are exceptionally useful for understanding nonlinear systems, but they also give us a nice way to understand stability, as in Figure equistab.1.

phase portrait

trajectories

3 These definitions of stability can be interpreted in terms of the free response of a system, as described, below.

Stability defined by the free response

4 Using the concept of the free response (no inputs, just initial conditions), we define the following types of stability for LTI systems[2].

2. N.S. Nise. Control Systems Engineering, 7th Edition. Wiley, 2015. ISBN: 9781118800829.

1. An LTI system is asymptotically stable if the free response approaches an

asymptotic stability

$\dot{x} = Ax$

$\begin{bmatrix}-2\\1\end{bmatrix} = A\begin{bmatrix}1\\1\end{bmatrix}$

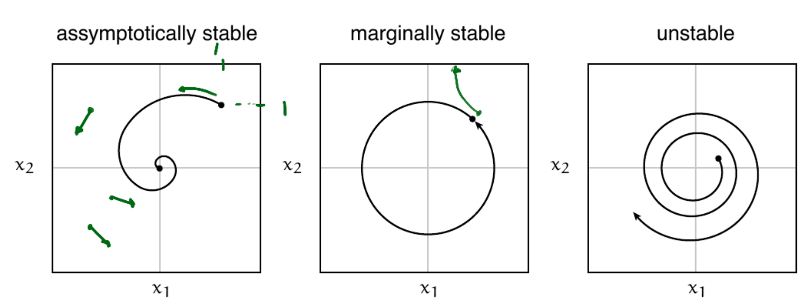

Figure equistab.1: a phase-portrait demonstration of (left) asymptotic stability, (center) marginal stability, and (right) instability for a second-order system.

   equilibrium state as time approaches infinity.
2. An LTI system is unstable if the free response grows without bound as time approaches infinity.
3. An LTI system is marginally stable if the free response neither decays nor grows but remains constant or oscillates as time approaches infinity.

instability

marginal stability

5 These statements imply that the free response alone governs stability. Recall that the free response y_{fr} of a system with characteristic equation roots λ_i with multiplicity m_i, for constants C_i, is

$$\boxed{y_{fr} = \sum_i C_i t^{m_i - 1} e^{\lambda_i t}}$$

Each term will either decay to zero, remain constant, or increase without bound—depending on the sign of the real part of the corresponding root of the characteristic equation $\mathrm{Re}(\lambda_i)$.

6 In other words, for an LTI system, the following statements hold.

$$\left.\begin{array}{l}
\text{1. An LTI system is asymptotically stable if, for all } \lambda_i,\ \mathrm{Re}(\lambda_i) < 0.\\
\text{2. An LTI system is unstable if, for any } \lambda_i,\ \mathrm{Re}(\lambda_i) > 0.\\
\text{3. An LTI system is marginally stable if,}\\
\quad \text{a) for all } \lambda_i,\ \mathrm{Re}(\lambda_i) \leqslant 0 \text{ and}\\
\quad \text{b) at least one } \mathrm{Re}(\lambda_i) = 0 \text{ and}\\
\quad \text{c) no } \lambda_i \text{ for which } \mathrm{Re}(\lambda_i) = 0 \text{ has multiplicity } m_i > 1.
\end{array}\right\} \text{based on free response}$$

Stability defined by the forced response

7 An alternate formulation of the stability definitions above is called the bounded-input bounded-output (BIBO) definition of stability, and states the following[3].

BIBO stability

3. Nise, Control Systems Engineering, 7th Edition.

1. A system is BIBO stable if every bounded input yields a bounded output.
2. A system is BIBO unstable if any bounded input yields an unbounded output.

BIBO stable

BIBO unstable

$y(t) = \int_0^t u(\tau)\,d\tau$

8 In terms of BIBO stability, marginal stability, then, means that a system has a bounded response to some inputs and an unbounded response to others. For instance, a second-order undamped system response to a sinusoidal input at the natural frequency is unbounded, whereas every other input yields a bounded output.

BIBO marginal stability

$u(t) = 1$

$\lim_{t \to \infty} y(t) = \lim_{t \to \infty} t = \infty$

BIBO unstable

9 Although we focus on the definitions of stability in terms of the free response, it is good to understand BIBO stability, as well.