# lti.vib Vibration isolation table analysis

1 In this example, we exercise many of the methods for modeling and analysis explored thus far.

2 Given the vibration isolation table model in Figure vib.1—with $m = 320$ kg, $k = 16000$ N/m, and $B = 1200$ N–m/s—derive:

1. a linear graph model,
2. a state-space model,
3. the equilibrium state $\bar{x}$ for the unit step input,
4. a transfer function model,
5. an input-output differential equation model with input $V_s$ and output $v_m$,
6. a solution for $v_m(t)$ for a unit step input $V_s(t) = 1$ m/s for $t \geqslant 0$,
7. the system's stability.

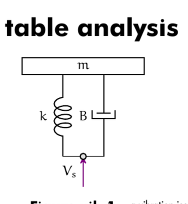


**Figure vib.1:** a vibration isolation table schematic with input velocity $V_s$.

## Linear graph and state-space models

3 The linear graph and normal tree are shown in Figure vib.2. Note that there is an equilibrium for this system, so we are justified in ignoring gravity and referencing any displacements to the static equilibrium position.[4] The state variables are the velocity of the mass $v_m$ and the force through the spring $f_k$ and the system order is $n = 2$. The input, state, and output vectors are

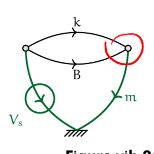


**Figure vib.2:** linear graph of the isolation table.

4. For a discussion of this ignoring of gravity, see Lec. lti.ghost.

$$u = \begin{bmatrix} V_s \end{bmatrix} \qquad x = \begin{bmatrix} v_m \\ f_k \end{bmatrix} \qquad y = \begin{bmatrix} v_m \end{bmatrix}.$$

The elemental equations are as follows.

| | |
|---|---|
| m | $\dot{v}_m = \frac{1}{m} f_m$ |
| k | $\dot{f}_k = k v_k$ |
| B | $f_B = B v_B$ |

The continuity and compatibility equations are as follows.

| branch | continuity equation |
|---|---|
| m | $f_m = f_k + f_B$ |

| link | compatibility equation |
|---|---|
| k | $v_k = V_s - v_m$ |
| B | $v_B = V_s - v_m$ |

The state equation can be found by substituting the continuity and compatibility equations into the elemental equations, and eliminating $f_B$, to yield

$$\dot{x} = \overbrace{\begin{bmatrix} -B/m & 1/m \\ -k & 0 \end{bmatrix}}^{A} x + \underbrace{\begin{bmatrix} B/m \\ k \end{bmatrix}}_{B} u \qquad (1a)$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \end{bmatrix} u. \qquad (1b)$$

## Equilibrium

4 Let's check to see if $A$ is invertible by trying to compute its inverse:

$$A^{-1} = \begin{bmatrix} -B/m & 1/m \\ -k & 0 \end{bmatrix}^{-1} \qquad (2)$$

$$= \frac{1}{k/m} \begin{bmatrix} 0 & -1/m \\ k & -B/m \end{bmatrix} \qquad (3)$$

So it has an inverse, after all! Let's use this to compute the equilibrium state:

$$\bar{x} = -A^{-1} B \bar{u} \qquad (4)$$

$$= \frac{-m}{k} \begin{bmatrix} 0 & -1/m \\ k & -B/m \end{bmatrix} \begin{bmatrix} B/m \\ k \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} \qquad (5)$$

$$= \frac{-m}{k} \begin{bmatrix} -k/m \\ 0 \end{bmatrix} \qquad (6)$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad (7)$$

So the system is in equilibrium with $v_m = 1$ m/s and $f_k = 0$ N. Since $v_m$ is also our output, we expect 1 m/s to be our steady-state output value.

## Transfer function model

5 The transfer function $H(s) = V_m(s)/V_s(s)$ will be used as a bridge to the input-output differential equation. The transfer function can be found from the usual formula, from Lecture ss.ss2tf2io,

$$H(s) = C(sI - A)^{-1} B + D. \qquad (8)$$

Let's first compute $(sI - A)^{-1}$:[5]

5. See ( Rowell and Wormley, System Dynamics: An Introduction, Sec. A.4.3) for details on the matrix inverse.

$$(sI - A)^{-1} = \left( \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -B/m & 1/m \\ -k & 0 \end{bmatrix} \right)^{-1} \qquad (9a)$$

$$= \begin{bmatrix} s + B/m & -1/m \\ k & s \end{bmatrix}^{-1} \qquad (9b)$$

$$= \frac{1}{(s + B/m)(s) - (-1/m)(k)} \begin{bmatrix} s & 1/m \\ -k & s + B/m \end{bmatrix} \qquad (9c)$$

$$= \frac{1}{s^2 + (B/m)s + k/m} \begin{bmatrix} s & 1/m \\ -k & s + B/m \end{bmatrix} \qquad (9d)$$

Now we're ready to compute the entirety of $H$:

$$H(s) = \frac{1}{s^2 + (B/m)s + k/m} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & 1/m \\ -k & s + B/m \end{bmatrix} \begin{bmatrix} B/m \\ k \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} \qquad (10a)$$

$$= \frac{1}{s^2 + (B/m)s + k/m} \begin{bmatrix} s & 1/m \end{bmatrix} \begin{bmatrix} B/m \\ k \end{bmatrix} \qquad (10b)$$

$$= \frac{(B/m)s + k/m}{s^2 + (B/m)s + k/m}. \qquad (10c)$$

## Input-output differential equation

6 The input-output differential equation can be found from the reverse of the procedure in Lecture ss.ss2tf2io. Beginning from the transfer function,

$$\frac{V_m(s)}{V_s(s)} = \frac{(B/m)s + k/m}{s^2 + (B/m)s + k/m} \Rightarrow \qquad (11a)$$

$$\left(s^2 + (B/m)s + k/m\right) V_m = \left((B/m)s + k/m\right) V_s \Rightarrow \qquad (11b)$$

$$\ddot{v}_m + (B/m)\dot{v}_m + (k/m)v_m = (B/m)\dot{V}_s + (k/m)V_s. \qquad (11c)$$

## Step response

7 The step response is found using superposition and the derivative property of LTI systems. The forcing function $f(t) = (B/m)\dot{V}_s + (k/m)V_s$ is composed of two terms, one of which has a derivative of the input $V_s$. Rather than attempting to solve the entire problem at once, we choose to find the response for a forcing function $f(t) = 1$ (for $t \geqslant 0$)—that is, the unit step response—and use superposition and the derivative property of LTI systems to calculate the composite response.

$\ddot{v}_m + \frac{B}{m} \dot{v}_m + \frac{k}{m} v_m = f(t) = 1$

### Unit step response

8 The characteristic equation of Equation 11c is

$$\lambda^2 + (B/m)\lambda + k/m = 0 \Rightarrow \qquad (12a)$$

$$= -\frac{B}{2m} \pm \frac{\sqrt{B^2 - 4mk}}{2m} \Rightarrow \qquad (12b)$$

$$\lambda_{1,2} = -1.875 \pm j6.818. \qquad (12c)$$

The roots are complex, so the system will have a damped sinusoidal step response. Let $\sigma = -1.875$ and $\omega = 6.818$ such that $\lambda_{1,2} = \sigma \pm j\omega$. The homogeneous solution is

$$v_{m_h}(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}. \qquad (13)$$

In this form, $C_1$ and $C_2$ are complex. It is somewhat easier to deal with

$$v_{m_h}(t) = C_1 e^{\sigma t} e^{j\omega t} + C_2 e^{\sigma t} e^{-j\omega t} \qquad (14a)$$

$$= e^{\sigma t} \left(C_1 \cos \omega t + jC_1 \sin \omega t + C_2 \cos \omega t - jC_2 \sin \omega t\right) \qquad (14b)$$

$$= e^{\sigma t} \left((C_1 + C_2) \cos \omega t + j(C_1 - C_2) \sin \omega t\right). \qquad (14c)$$

Let $C_3 = C_1 + C_2$ and $C_4 = j(C_1 - C_2)$ such that

$$v_{m_h}(t) = e^{\sigma t} \left(C_3 \cos \omega t + C_4 \sin \omega t\right). \qquad (15)$$

This is a decaying (because $\sigma < 0$) sinusoid. A nice aspect of this new form is that $C_3$ and $C_4$ are real.

9 Now, the particular solution can be found by assuming a solution of the form $v_{m_p}(t) = K$ for $t \geqslant 0$. Substituting this into Equation 11c (with forcing $f(t) = 1$, we attempt to find a solution for $K$ (that is, determine it):

$$(k/m)K = 1 \Rightarrow K = m/k. \qquad (16)$$

Therefore, $v_{m_p}(t) = m/k$ is a solution, and therefore the general solution is

$$v_{m_{step}}(t) = v_{m_h}(t) + v_{m_p}(t) \qquad (17a)$$

$$= e^{\sigma t} \left(C_3 \cos \omega t + C_4 \sin \omega t\right) + m/k. \qquad (17b)$$

This leaves the specific solution, to be found applying the initial conditions (assumed to be zero). Before we do so, however, we need the time-derivative of the $v_{m_{step}}$:

$$\dot{v}_{m_{step}}(t) = e^{\sigma t} \left((C_3\sigma + C_4\omega) \cos(\omega t) + (C_4\sigma - C_3\omega) \sin(\omega t)\right). \qquad (18)$$

Now, applying the initial conditions,

$$v_{m_{step}}(0) = 0 \Rightarrow \qquad (19a)$$

$$C_3 = -m/k \qquad (19b)$$

$$\dot{v}_{m_{step}}(0) = 0 \Rightarrow 0 = C_3\sigma + C_4\omega \Rightarrow \qquad (19c)$$

$$C_4 = \frac{\sigma}{\omega} \cdot \frac{m}{k}. \qquad (19d)$$

10 It's good form to re-write this as a single sinusoid:

$$v_{m_{step}}(t) = v_{m_h}(t) + v_{m_p}(t) \qquad (20a)$$

$$= A_1 e^{\sigma t} \cos(\omega t + \psi_1) + m/k \qquad (20b)$$

where we have used Lecture math.trig to find

$$A_1 = \sqrt{C_3^2 + C_4^2} \qquad (21a)$$

$$\psi_1 = -\arctan(C_4/C_3). \qquad (21b)$$

### Superposition and the derivative property

11 Recall that the actual forcing function is a linear combination of the input and its time-derivative. Therefore, it is expedient to re-write the time-derivative of the unit step response:

$$\dot{v}_{m_{step}}(t) = A_1 e^{\sigma t} \left(\sigma \cos(\omega t + \psi_1) - \omega \sin(\omega t + \psi_1)\right) \qquad (22a)$$

$$= A_1 A_2 e^{\sigma t} \cos(\omega t + \psi_1 + \psi_2) \qquad (22b)$$

where

$$A_2 = \sqrt{\sigma^2 + \omega^2} \qquad (22c)$$

$$\psi_2 = -\arctan(-\omega/\sigma). \qquad (22d)$$

Finally, applying superposition and the derivative rule of LTI systems,

$$v_m(t) = (B/m)\dot{v}_{m_{step}}(t) + (k/m)v_{m_{step}} \qquad (23a)$$

$$= \frac{B}{m} A_1 A_2 e^{\sigma t} \cos(\omega t + \psi_1 + \psi_2) + \frac{k}{m} A_1 e^{\sigma t} \cos(\omega t + \psi_1) + 1. \qquad (23b)$$

This is the solution!

12 It's worth plotting the response. Begin by defining the system parameters.

```
m = 320; % kg ... mass
k = 16000; % N/m ... spring constant
B = 1200; % N-m/s ... damping coefficient
```

Now define the secondary parameters.

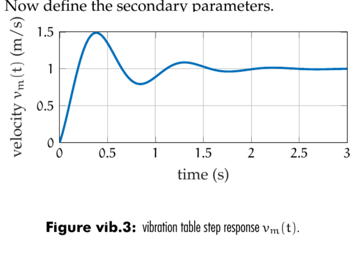


**Figure vib.3:** vibration table step response $v_m(t)$.

```
lambda = -B/(2*m)+[-1,1]*sqrt(B^2-4*m*k)/(2*m);
sigma = real(lambda(1));
omega = imag(lambda(2));
K = m/k;
C3 = -m/k;
C4 = sigma/omega*m/k;
A1 = sqrt(C3^2+C4^2);
psi1 = -atan2(C4,C3);
A2 = sqrt(sigma^2+omega^2);
psi2 = -atan2(-omega,sigma);
```

Finally, the solution for $v_m(t)$ can be defined as an anonymous function.

```
vm = @(t) ...
    A1*A2*B/m*exp(sigma*t).*cos(omega*t+psi1+psi2)+...
    A1*k/m*exp(sigma*t).*cos(omega*t+psi1)+...
    1;
```

Now, plot over the first few seconds. The results are shown in Figure vib.3.

```
t_a = linspace(0,3,200);
h = figure;
p = plot(t_a,vm(t_a),'linewidth',1.5);
xlabel('time (s)')
ylabel('velocity $v_m(t)$ (m/s)',...
    'interpreter','latex');
grid on
hgsave(h,'figures/temp');
```

13 Note that the steady-state output value agrees with that predicted by the equilibrium analysis, above.

## Stability

14 We have learned what we need in order to analyze the system's stability. The roots of the characteristic equation were $\lambda_{1,2} = -1.875 \pm j6.818$, which clearly all have negative real parts, and therefore the system is asymptotically stable.