ssresp.diag Diagonalizing basis

1 It is useful to transform a system's state vector $\mathbf{x}$ into a special basis that diagonalizes—leaves nonzero components along only the diagonal—the system's A-matrix. For systems with n distinct eigenvalues, to which we limit ourselves in this discussion,[4] this is always possible. In diagonalized form, it will be relatively easy to solve for the state transition matrix Φ.

diagonalizes

4. See Appendix adv.eig for general considerations.

Changing basis in the state equation

2 As with all basis transformations, the basis transformation we seek can be written

$$\mathbf{x} = P\mathbf{x}' \quad \Rightarrow \mathbf{x}' = P^{-1}\mathbf{x}, \tag{1}$$

where P is the transformation matrix, $\mathbf{x}$ is a representation of the state vector in the original basis, and $\mathbf{x}'$ is a representation of the state vector in the new basis.[5]

transformation matrix

5. We are being a bit fast-and-loose with terminology here: a vector is an object that does not change under basis transformation, only its components and basis vectors do. However, we use the common notational and terminological abuses.

3 Substituting this transformation into the standard linear state-model equations yields the model

$$\dot{\mathbf{x}}' = \underbrace{P^{-1}AP}_{A'}\mathbf{x}' + \underbrace{P^{-1}B}_{B'}\mathbf{u} \tag{2a}$$

$$\mathbf{y} = \underbrace{CP}_{C'}\mathbf{x}' + \underbrace{D}_{D'}\mathbf{u}. \tag{2b}$$

Modal and eigenvalue matrices

4 Let a state equation have matrix A with n distinct eigenvalues (λ_i) and eigenvectors $(\mathbf{m}_i)$. Let the eigenvalue matrix Λ be defined as

eigenvalue matrix

$$\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix}$$

5 Furthermore, let the modal matrix M be defined as

modal matrix

$$M = \begin{bmatrix} | & | & & | \\ \mathbf{m}_1 & \mathbf{m}_2 & \cdots & \mathbf{m}_n \\ | & | & & | \end{bmatrix} \tag{3}$$

Diagonalization of the state equation

6 Let the modal matrix M be the transformation matrix for our state-model. Then[6] $\mathbf{x}' = M^{-1}\mathbf{x}$.

6. As long as there are n distinct eigenvalues, M is invertible.

7 The state equation becomes

$$\dot{\mathbf{x}}' = \underbrace{M^{-1}AM}_{A'}\mathbf{x}' + M^{-1}B\mathbf{u}. \tag{4}$$

The eigenproblem implies that

$$A[m_1 \; m_2 \cdots m_n] = [m_1 \; m_2 \cdots m_n]\Lambda$$
$$AM = M\Lambda$$
$$M^{-1}AM = M^{-1}M\Lambda = \Lambda$$

That is, $A' = \Lambda$! Recall that Λ is diagonal; therefore, we have diagonalized the state-space model. In full-form, the diagonalized model is

diagonalized

$$\dot{\mathbf{x}}' = \underbrace{\Lambda}_{A'}\mathbf{x}' + \underbrace{M^{-1}B}_{B'}\mathbf{u} \tag{5a}$$

$$\mathbf{y} = \underbrace{CM}_{C'}\mathbf{x}' + \underbrace{D}_{D'}\mathbf{u}. \tag{5b}$$

Computing the state transition matrix

8 Recall our definition of the state transition matrix $\Phi(t) = e^{At}$. Directly applying this to the diagonalized system of Eq. 5,

$$\Phi'(t) = e^{\Lambda t} \tag{6a}$$

$$= \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix}. \tag{6b}$$

In this last equality, we have used the diagonal property of the state transition matrix, defined in Lec. ssresp.response.

diagonal property

9 Recall that the free response solution to the state equation is given by the initial condition and state transition matrix, so

$$\Phi'(t)\,x'(0) = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & e^{\lambda_2 t} & \\ 0 & & \ddots \; e^{\lambda_n t} \end{bmatrix}\begin{bmatrix} x_1'(0) \\ x_2'(0) \\ \vdots \\ x_n'(0) \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t}x_1'(0) \\ e^{\lambda_2 t}x_2'(0) \\ \vdots \\ e^{\lambda_n t}x_n'(0) \end{bmatrix} = x'_{fr}(t)$$

$$\mathbf{x}'_{\text{fr}}(t) = \Phi'(t)\mathbf{x}'(0) \tag{7a}$$

$$= x_1'(0)e^{\lambda_1 t} + x_2'(0)e^{\lambda_2 t} + \cdots + x_n'(0)e^{\lambda_n t} \tag{7b}$$

incorrect

where the initial conditions are $\mathbf{x}'(0) = M^{-1}\mathbf{x}(0)$. We have completely decoupled each state's free response, one of the remarkable qualities of the diagonalized system.

10 At this point, one could simply solve the diagonalized system for $\mathbf{x}'(t)$, then convert the solution to the original basis with $\mathbf{x}(t) = M\mathbf{x}'(t)$.

11 Sometimes, we might prefer to transform the diagonalized-basis state transition matrix into the original basis. The following is a derivation of that transformation.

12 Beginning with the free response solution in the diagonalized-basis and transforming the equation into the original basis, we find an expression for the original state transition matrix, as follows.

$$x'_{fr}(t) = \Phi'(t)\,x'(0)$$
$$M^{-1}x_{fr}(t) = \Phi'(t)\,M^{-1}x(0)$$
$$x_{fr}(t) = M\,\Phi'(t)\,M^{-1}x(0)$$

$$x_{fr}(t) = \Phi(t)\,x(0)$$

This last expression is just the free response solution in the original basis, so we can identify

$$\Phi(t) = M\Phi'(t)M^{-1}. \tag{8}$$

This is a powerful result. Eq. 8 is the preferred method of deriving the state transition matrix for a given system. The eigenvalues give Φ' and the eigenvectors give M.

Example ssresp.diag−1 **re: state free response**

For the state equation

$$\dot{\mathbf{x}} = \begin{bmatrix} -2 & 2 \\ 2 & -3 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}\mathbf{u}$$

find the state's free response to initial condition $\mathbf{x}(0) = \begin{bmatrix} 2 & -1 \end{bmatrix}^\top$.

$$A = \begin{bmatrix} -2 & 2 \\ 2 & -3 \end{bmatrix}$$

$$x_{fr}(t) = \Phi(t)\,x(0)$$

$$\Phi'(t) = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$$

$$M\,\Phi'(t)\,M^{-1} = \Phi(t)$$

$$x_{fr}(t) = \begin{bmatrix} -0.62 & -0.79 \\ 0.79 & -0.62 \end{bmatrix}\begin{bmatrix} e^{-4.6t} & 0 \\ 0 & e^{-0.44t} \end{bmatrix}\begin{bmatrix} -0.62 & 0.79 \\ -0.79 & -0.62 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$y(t) = Cx(t) + \cancel{Du(t)}$$