RW 10.3

Suppose a linear system has $\Phi(t)$

a) Show that if $\Phi(t)$ is evaluated at $t=T$ the response at $t=nT$ integer n is

$$x(nT) = (\Phi(T))^n x(0)$$

$$\Phi(T_1 + T_2) = \Phi(T_1)\Phi(T_2)$$

$$\Phi(2T) = \Phi(T)\Phi(T)$$

$$\Phi\left(\sum_{i=0}^{n} T_i\right) = \prod_{i=0}^{n} \Phi(T_i) \qquad T_i \text{ is constant for all } i$$

$$\Phi(nT) = (\Phi(T))^n$$

$$\Phi(nT) = \Phi(t)$$

$$x(t) = \Phi(t)\, x(0)$$

$$x(nT) = (\Phi(T))^n x(0)$$

b) if $x(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ $\quad \Phi(0.1) = \begin{bmatrix} 0.3 & 0.5 \\ 0 & 0.4 \end{bmatrix}$

find $x(0.1)$ and $x(0.3)$

$T = 0.1 \qquad t = 0.1 \qquad n = 1$

$$x(nT) = (\Phi(0.1))^1 x(0)$$

$$x(0.1) = \begin{bmatrix} 0.3 & 0.5 \\ 0 & 0.4 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0.3(2) + 0.5(-1) \\ -0.4 \end{bmatrix}$$

$$\boxed{= \begin{bmatrix} 0.1 \\ -0.4 \end{bmatrix}}$$

$T = 0.1 \qquad n = 3 \qquad t = 0.3$

$$x(0.3) = (\Phi(0.1))^3 x(0)$$

$$= (\Phi(0.1))^2 \underbrace{\Phi(0.1)\, x(0)}_{x(0.1)}$$

$$= \begin{bmatrix} 0.3 & 0.5 \\ 0 & 0.4 \end{bmatrix}\begin{bmatrix} 0.3 & 0.5 \\ 0 & 0.4 \end{bmatrix}\begin{bmatrix} 0.1 \\ -0.4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.3 & 0.5 \\ 0 & 0.4 \end{bmatrix}\begin{bmatrix} 0.1(0.3) - (0.5)(0.4) \\ -0.4(0.4) \end{bmatrix} =$$

$$= \begin{bmatrix} 0.3 & 0.5 \\ 0 & 0.4 \end{bmatrix}\begin{bmatrix} -0.17 \\ -0.16 \end{bmatrix}$$

$$= \begin{bmatrix} -0.17(0.3) - 0.5(0.16) \\ -0.4(0.16) \end{bmatrix} = \boxed{\begin{bmatrix} -0.13 \\ -0.064 \end{bmatrix}}$$

c) Show $\Phi(T_1 + T_2) = \Phi(T_1)\Phi(T_2)$

$$M^{-1}M\, \Phi'(T_1 + T_2)\, M^{-1}M = M^{-1}M\, \Phi'(T_1)\, \cancel{M^{-1}M}\, \Phi'(T_2)\, M^{-1}M$$

$$\Phi'(T_1 + T_2) = \Phi'(T_1)\Phi'(T_2)$$

$$\begin{bmatrix} e^{\lambda_1(T_1+T_2)} & & 0 \\ & e^{\lambda_2(T_1+T_2)} & \\ 0 & & \ddots \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 T_1} & & 0 \\ & e^{\lambda_2 T_1} & \\ 0 & & \ddots \end{bmatrix}\begin{bmatrix} e^{\lambda_1 T_2} & & 0 \\ & e^{\lambda_2 T_2} & \\ 0 & & \ddots \end{bmatrix}$$

$$e^{\lambda_i(T_1+T_2)} = e^{\lambda_i T_1} e^{\lambda_i T_2}$$

$$= e^{\lambda_i T_1 + \lambda_i T_2}$$

$$= e^{\lambda_i(T_1+T_2)} \quad \square$$