four.fsexa Complex Fourier series example

1 There are several flavors of Fourier series problem: trigonometric/exponential, analysis/synthesis, plotting partial sums/plotting spectra. Of course, problems just present us an opportunity to traverse part of the landscape (to mix two metaphors like 31 similies).

Example four.fsexa−1

2 Consider a recified sinusoid

$$f(t) = |A\cos(\omega t)|$$

for $A, \omega, t \in \mathbb{R}$, shown in Fig. fsexa.1. The fundamental period is $T = \pi/\omega$, half the unrectified period.

a. Perform a complex Fourier analysis on $f(t)$, computing the complex Fourier components $c_{\pm n}$.
b. Compute and plot the magnitude and phase spectra.
c. Convert $c_{\pm n}$ to trigonometric components a_n and b_n.

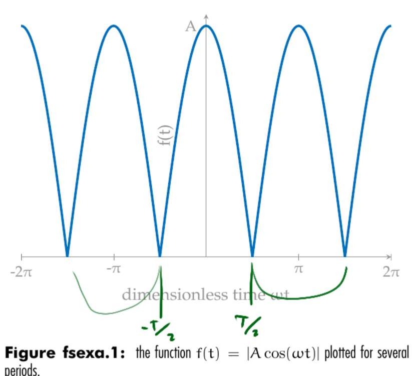


Figure fsexa.1: the function $f(t) = |A\cos(\omega t)|$ plotted for several periods.

Part a: complex Fourier analysis

3 The complex Fourier analysis of Definition four.2 will be applied in a moment. However, it is convenient to first convert f into an exponential. We can write f over a single period $t \in [-T/2, T/2)$ as

$$|A\cos(\omega t)| = |A||\cos(\omega t)|$$
(absolute value property)
$$= |A|\cos(\omega t)$$
(already positive)
$$= |A|\tfrac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right)$$
(Euler, Eq. 2)

4 Applying Fourier analysis à la Definition four.2 with harmonic frequency $\omega_n = 2\pi n/T$,

$$\begin{aligned}
c_{\pm n} &= \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-j\omega_n t}\,dt \\
&= \frac{1}{T}\int_{-T/2}^{T/2} |A|\frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right)e^{-j\omega_n t}\,dt \\
&= \frac{|A|}{2T}\int_{-T/2}^{T/2} \left(e^{j\omega t} + e^{-j\omega t}\right)e^{-j\omega_n t}\,dt \\
&= \frac{|A|}{2T}\int_{-T/2}^{T/2} \left(e^{j(\omega-\omega_n)t} + e^{-j(\omega+\omega_n)t}\right)dt \\
&= \frac{|A|}{2T}\left(\frac{1}{j(\omega-\omega_n)}e^{j(\omega-\omega_n)t} - \frac{1}{j(\omega+\omega_n)}e^{-j(\omega+\omega_n)t}\right)\Bigg|_{-T/2}^{T/2} \\
&= \frac{|A|}{2T}\left(\frac{1}{j(\omega-\omega_n)}e^{j(\omega-\omega_n)T/2} - \frac{1}{j(\omega+\omega_n)}e^{-j(\omega+\omega_n)T/2} + \right. \\
&\quad \left. - \frac{1}{j(\omega-\omega_n)}e^{-j(\omega-\omega_n)T/2} + \frac{1}{j(\omega+\omega_n)}e^{j(\omega+\omega_n)T/2}\right) \\
&= \frac{|A|}{j2T(\omega-\omega_n)}\left(e^{j(\omega-\omega_n)T/2} - e^{-j(\omega-\omega_n)T/2}\right) + \\
&\quad + \frac{|A|}{j2T(\omega+\omega_n)}\left(e^{j(\omega+\omega_n)T/2} - e^{-j(\omega+\omega_n)T/2}\right) \\
&= \frac{|A|}{T(\omega-\omega_n)}\sin((\omega-\omega_n)T/2) + \frac{|A|}{T(\omega+\omega_n)}\sin((\omega+\omega_n)T/2).
\end{aligned}$$

5 This can be simplified further if we substitute $T = \pi/\omega$ and $\omega_n = 2\pi n/T = 2n\omega$,

$$c_{\pm n} = \frac{|A|}{\pi(1-2n)}\sin\left(\frac{(1-2n)\pi}{2}\right) + \frac{|A|}{\pi(1+2n)}\sin\left(\frac{(1+2n)\pi}{2}\right)$$

6 Using a product-to-sum trigonometric identity (Appendix math.trig), this further simplifies to

$$c_{\pm n} = \frac{-2|A|}{\pi(4n^2-1)}\cos(\pi n),$$

which, for n odd or even,

$$c_{\pm n} = \begin{cases} \frac{2|A|}{\pi(4n^2-1)} & n \text{ odd} \\ \frac{-2|A|}{\pi(4n^2-1)} & n \text{ even.} \end{cases}$$

7 Alternatively we could use Matlab's Symbolic Math Toolbox rather straightforwardly.

```
syms A n w wn T t 'real' % symbolic, real
```

8 Now define the function of time f and the known relations in a dictionary.

```
f = abs(A)*cos(w*t);
props.T = pi/w;
props.wn = 2*n*w;
```

9 Now apply the same Fourier analysis as before.

```
c_n1 = 1/T*int(f*exp(-j*wn*t),t,-T/2,T/2);
c_n = simplify(subs(c_n1,props))
```

```
c_n =

-(2*cos(pi*n)*abs(A))/(pi*(4*n^2 - 1))
```

10 Nice! This is the same result. We can even check our odd/even assumptions.

```
assume((n-1)/2,'integer') % odd
simplify(c_n)
assume(n,'clear') % clear assumptions
assume(n/2,'integer') % even
simplify(c_n)
assume(n,'clear') % clear before moving on
assume(n,'real')
```

```
ans =

(2*abs(A))/(pi*(4*n^2 - 1))


ans =

-(2*abs(A))/(pi*(4*n^2 - 1))
```

11 These are also what we got before.

Parb b: harmonic amplitude and phase with spectra

12 According to Eq. 11, the harmonic amplitude is

$$\begin{aligned}
C_n &= 2\sqrt{c_n c_{-n}} \\
&= \frac{4|A|}{\pi|4n^2-1|}|\cos(\pi n)|
\end{aligned}$$

13 Let's check with Matlab.

```
assume(n,'real');
C_n = simplify(2*sqrt(c_n*subs(c_n,n,-n)))
assume(n,'clear');
assume(n,'integer');
C_n = simplify(2*sqrt(c_n*subs(c_n,n,-n)))
```

```
C_n =

(4*abs(A)*abs(cos(pi*n)))/(pi*abs(4*n^2 - 1))


C_n =

(4*abs(A))/(pi*abs(4*n^2 - 1))
```

14 We see that if we assume n is an integer, C_n simplifies even further than we took it by-hand.

15 Plotting the harmonic amplitude is straightforward. First make C_n something that can be numerically evaluated and choose parameters.

```
p.A = 1;
C_n_fun = matlabFunction( ...
  subs(C_n, p) ...
);
```

16 Now we plot.

```
n_a = -10:10;
figure
stem(n_a,C_n_fun(n_a))
xlabel('\pm n')
ylabel('harmonic amplitude C_n/|A|')
```

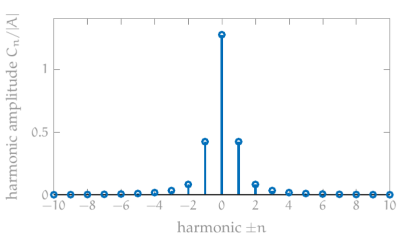


Figure fsexa.2: the harmonic amplitude C_n.

17 Let's find the phase à la Eq. 12 with Matlab directly.

```
phase_n = simplify(atan2(imag(c_n),real(c_n)))
```

```
phase_n =

(pi*sign(((-1)^n*abs(A))/(4*n^2 -
↪  1))*(sign(((-1)^n*abs(A))/(4*n^2 - 1)) + 1))/2
```

18 The sign function just returns the sign of its argument. It's difficult to see, but this expression only takes on the following two values:

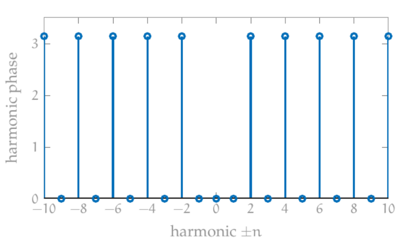
0, π

19 We can plot the phase similarly to how we plotted the amplitude. First we get a numerically evaluable function.

```
phase_n_fun = matlabFunction( ...
  subs(phase_n, p) ...
);
```

20 Now we plot.

```
figure
stem(n_a,phase_n_fun(n_a))
xlabel('\pm n')
ylabel('harmonic phase')
```



Part c: conversion to trig form

21 According to Definition four.3, the trigonometric components can be computed from the complex components as follows.

```
a_n = simplify(c_n + subs(c_n,n,-n))
b_n = simplify(j*(c_n - subs(c_n,n,-n)))
```

```
a_n =

-(4*(-1)^n*abs(A))/(pi*(4*n^2 - 1))


b_n =

0
```

22 The fact that $b_n = 0$ should not surprise us: $f(t)$ is even after all!