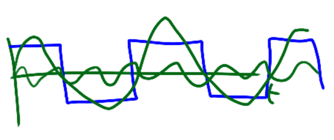
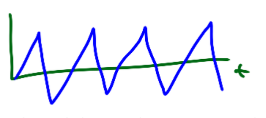
four.series Fourier series

1 Fourier series are mathematical series that can represent a periodic signal as a sum of sinusoids at different amplitudes and frequencies. They are useful for solving for the response of a system to periodic inputs. However, they are probably most important conceptually: they are our gateway to thinking of signals in the frequency domain—that is, as functions of frequency (not time). To represent a function as a Fourier series is to analyze it as a sum of sinusoids at different frequencies[1] ω_n and amplitudes a_n. Its frequency spectrum is the functional representation of amplitudes a_n versus frequency ω_n.



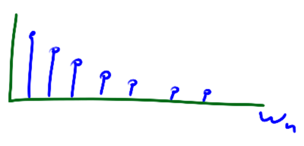
frequency domain



Fourier analysis

1. It's important to note that the symbol ω_n, in this context, is not the natural frequency, but a frequency indexed by integer n.

frequency spectrum



2 Let's begin with the definition.

Definition four.1: Fourier series: trigonometric form

The Fourier analysis of a periodic function $y(t)$ is, for $n \in \mathbb{N}_0$, period T, and angular frequency $\omega_n = 2\pi n/T$,

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} y(t)\cos(\omega_n t)dt \tag{1}$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} y(t)\sin(\omega_n t)dt. \tag{2}$$

The Fourier synthesis of a periodic function $y(t)$ with analysis components a_n and b_n corresponding to ω_n is

$$y(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(\omega_n t) + b_n\sin(\omega_n t). \tag{3}$$

3 Let's consider the complex form of the Fourier series, which is analogous to Definition four.1. It may be helpful to review Euler's formula(s) – see Appendix com.euler.

Definition four.2: Fourier series: complex form

The Fourier analysis of a periodic function $y(t)$ is, for $n \in \mathbb{N}_0$, period T, and angular frequency $\omega_n = 2\pi n/T$,

$$c_{\pm n} = \frac{1}{T}\int_{-T/2}^{T/2} y(t)e^{-j\omega_n t}dt. \tag{4}$$

The Fourier synthesis of a periodic function $y(t)$ with analysis components c_n corresponding to ω_n is

$$y(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\omega_n t}. \tag{5}$$

4 We call the integer n a harmonic and the frequency associated with it,

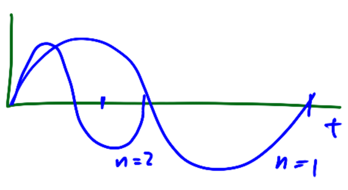
harmonic

$$\omega_n = 2\pi n/T, \tag{6}$$

the harmonic frequency. There is a special name for the first harmonic ($n = 1$): the fundamental frequency. It is called this because all other frequency components are integer multiples of it.

harmonic frequency

fundamental frequency



5 It is also possible to convert between the two representations above.

Definition four.3: Fourier series: converting between forms

The complex Fourier analysis of a periodic function $y(t)$ is, for $n \in \mathbb{N}_0$ and a_n and b_n as defined above,

$$c_{\pm n} = \frac{1}{2}\left(a_{|n|} \mp jb_{|n|}\right) \tag{7}$$

The sinusoidal Fourier analysis of a periodic function $y(t)$ is, for $n \in \mathbb{N}_0$ and c_n as defined above,

$$a_n = c_n + c_{-n} \text{ and} \tag{8}$$

$$b_n = j\left(c_n - c_{-n}\right). \tag{9}$$

6 The harmonic amplitude C_n is

harmonic amplitude

$$C_n = \sqrt{a_n^2 + b_n^2} \tag{10}$$

$$= 2\sqrt{c_n c_{-n}}. \tag{11}$$

A magnitude line spectrum is a graph of the harmonic amplitudes as a function of the harmonic frequencies. The harmonic phase is

magnitude line spectrum

harmonic phase

$$\theta_n = -\operatorname{arctan}_2(b_n, a_n) = -\arctan\left(\frac{a_n}{b_n}\right)$$

(see Appendix math.trig)

$$= \operatorname{arctan}_2(\operatorname{Im}(c_n), \operatorname{Re}(c_n)). \tag{12}$$

7 The illustration of Fig. series.1 shows how sinusoidal components sum to represent a square wave. A line spectrum is also shown.

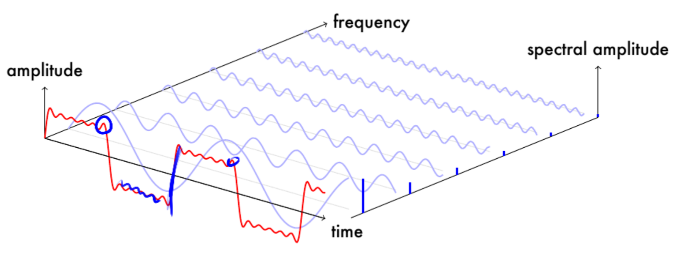
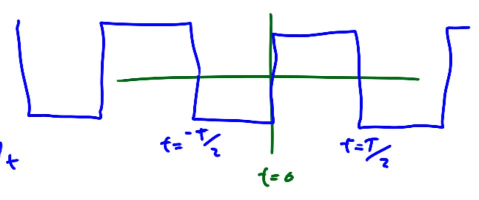


Figure series.1: a partial sum of Fourier components of a square wave shown through time and frequency. The spectral amplitude shows the amplitude of the corresponding Fourier component.

8 Let us compute the associated spectral components in the following example.

Example four.series–1 **re: Fourier series analysis: line spectrum**

Compute the first five harmonic amplitudes that represent the line spectrum for a square wave in the figure above.



Assume amplitude of 1

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} y(t)\cos(2\pi n t/T)\,dt$$

$$= \frac{2}{T}\int_{-T/2}^{0}(-1)\cos(2\pi n t/T)\,dt + \frac{2}{T}\int_{0}^{T/2}\cos(2\pi n t/T)\,dt$$

$\cos(-a) = \cos(a)$

$$= 0$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} y(t)\sin(2\pi n t/T)\,dt$$

$$= \frac{2}{T}\int_{-T/2}^{0}(-1)\sin(2\pi n t/T)\,dt + \frac{2}{T}\int_{0}^{T/2}\sin(2\pi n t/T)\,dt$$

$$= +\frac{\cancel{2}}{\cancel{T}}\frac{\cancel{T}}{\cancel{2}\pi n}\cos(2\pi n t/T)\Big|_{-T/2}^{0} - \frac{\cancel{2}}{\cancel{T}}\frac{\cancel{T}}{\cancel{2}\pi n}\cos(2\pi n t/T)\Big|_{0}^{T/2}$$

$$= \frac{1}{\pi n}\left(1 - \cos\left(\cancel{2}\pi n\left(\frac{-\cancel{T}}{\cancel{2}}\right)\frac{1}{\cancel{T}}\right)\right) - \frac{1}{\pi n}\left(\cos\left(\cancel{2}\pi n\left(\frac{\cancel{T}}{\cancel{2}}\right)\frac{1}{\cancel{T}}\right) - 1\right)$$

$$= \frac{1}{\pi n}\left(1 - \cos(\pi n) - \cos(\pi n) + 1\right)$$

$$= \frac{2}{\pi n}\left(1 - \cos(\pi n)\right)$$

$n=0 \quad \cos(\pi n) = 1$
$n=1 \quad \cos(\pi n) = -1$
$n=2 \quad \cos(\pi n) = 1$
$n=3 \quad \cos(\pi n) = -1$

$$b_n = \begin{cases} 0 & n \text{ even} \\ \frac{4}{n\pi} & n \text{ odd} \end{cases}$$

$$C_n = \sqrt{a_n^2 + b_n^2}$$

$$= b_n$$

$C_0 = 0$

$C_1 = \frac{4}{\pi}$

$C_2 = 0$

$C_3 = \frac{4}{3\pi}$

$C_4 = 0$

$C_5 = \frac{4}{5\pi}$