freq.fir Frequency and impulse response

1 This lecture proceeds in three parts. First, the Fourier transform is used to derive the frequency response function. Second, this is used to derive the frequency response. Third, the frequency response for an impulse input is explored.

Frequency response functions

2 Consider a dynamic system described by the input-output differential equation—with variable y representing the output, dependent variable time t, variable u representing the input, constant coefficients a_i, b_j, order n, and $m \leqslant n$ for $n \in \mathbb{N}_0$—as:

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y = b_m\frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1}u}{dt^{m-1}} + \cdots + b_1\frac{du}{dt} + b_0 u. \tag{1}$$

3 The Fourier transform $\mathcal{F}$ of Eq. 1 yields something interesting (assuming zero initial conditions):

Fourier transform

$$\begin{aligned}
\mathcal{F}\left(\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y\right) &= \\
\mathcal{F}\left(b_m\frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1}u}{dt^{m-1}} + \cdots + b_1\frac{du}{dt} + b_0 u\right) &\Rightarrow \\
\mathcal{F}\left(\frac{d^n y}{dt^n}\right) + a_{n-1}\mathcal{F}\left(\frac{d^{n-1}y}{dt^{n-1}}\right) + \cdots + a_1\mathcal{F}\left(\frac{dy}{dt}\right) + a_0\mathcal{F}(y) &= \\
b_m\mathcal{F}\left(\frac{d^m u}{dt^m}\right) + b_{m-1}\mathcal{F}\left(\frac{d^{m-1}u}{dt^{m-1}}\right) + \cdots + b_1\mathcal{F}\left(\frac{du}{dt}\right) + b_0\mathcal{F}(u) &\Rightarrow \\
(j\omega)^n Y + a_{n-1}(j\omega)^{n-1}Y + \cdots + a_1(j\omega)Y + a_0 Y &= \\
b_m(j\omega)^m U + b_{m-1}(j\omega)^{m-1}U + \cdots + b_1(j\omega)U + b_0 U.
\end{aligned}$$

Solving for Y,

$$\boxed{Y = \frac{b_m(j\omega)^m + b_{m-1}(j\omega)^{m-1} + \cdots + b_1(j\omega) + b_0}{(j\omega)^n + a_{n-1}(j\omega)^{n-1} + \cdots + a_1(j\omega) + a_0} U}$$

The inverse Fourier transform $\mathcal{F}^{-1}$ of Y is the forced response. However, this is not our primary concern; rather, we are interested to solve for the frequency response function $H(j\omega)$ as the ratio of the output transform Y to the input transform U, i.e.[1]

forced response

$$H(j\omega) \equiv \frac{Y(\omega)}{U(\omega)} \tag{2a}$$

$$= \frac{b_m(j\omega)^m + b_{m-1}(j\omega)^{m-1} + \cdots + b_1(j\omega) + b_0}{(j\omega)^n + a_{n-1}(j\omega)^{n-1} + \cdots + a_1(j\omega) + a_0}. \tag{2b}$$

1. It is traditional to use the non-standard, single-sided Fourier transform for the frequency response function for $H(j\omega)$. The motivation is that it then pairs well with the (single-sided) Laplace transform's transfer function.

4 Note that a frequency response function can be converted to a transfer function via the substitution $j\omega \mapsto s$ and, conversely, a transfer function can be converted to a frequency response function[2] via the substitution $s \mapsto j\omega$, as in

$$\boxed{H(j\omega) = H(s)\big|_{s \to j\omega}}$$

2. A caveat is that $H(j\omega) = H(s)|_{s\mapsto j\omega}$ only holds if the corresponding single-sided Fourier transform exists.

$$SS \longrightarrow C(sI - A)^{-1}B + D$$
$$\downarrow$$
$$H(s)$$
$$\downarrow$$
$$H(j\omega) \longleftarrow s \to j\omega$$

It is often easiest to first derive a transfer function—using any of the methods described, previously—then convert this to a frequency response function.

Frequency response

5 From above, we can solve for the output response y from the frequency response function by taking the inverse Fourier transform:

$$y(t) = \mathcal{F}^{-1}Y(\omega). \tag{3}$$

From the definition of the frequency response function (2a),

$$y(t) = \mathcal{F}^{-1}(H(j\omega)U(\omega)). \tag{4}$$

6 The convolution theorem states that, for two functions of time h and u,

convolution theorem

$$\mathcal{F}(h * u) = \mathcal{F}(h)\mathcal{F}(u) \tag{5a}$$

$$= H(j\omega)U(\omega), \tag{5b}$$

where the convolution operator $*$ is defined by

convolution operator $*$

$$(h * u)(t) \equiv \int_{-\infty}^{\infty} h(\tau)u(t-\tau)\, d\tau. \tag{6}$$

Therefore,

$$x_{f_0} = D\,u(t) + \int_0^t \Phi(\tau)\, u(t-\tau)\, d\tau$$

$$\boxed{\begin{aligned} y(t) &= \mathcal{F}^{-1}(H(j\omega)U(\omega)) \\ &= (h * u)(t) && \text{(from (5b))} \\ &= \int_{-\infty}^{\infty} h(\tau)u(t-\tau)\,d\tau && \text{(from (6))} \end{aligned}}$$

This is the frequency response in terms of all time-domain functions.

frequency response

Impulse response

7 The frequency response result includes an interesting object: $h(t)$. What is the physical significance of h, other than its definition, as the inverse Fourier transform of $H(j\omega)$?

8 Consider the singularity input $u(t) = \delta(t)$, an impulse. The frequency response is

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\delta(t-\tau)\, d\tau. \tag{7}$$

The so-called sifting property of δ yields

sifting property

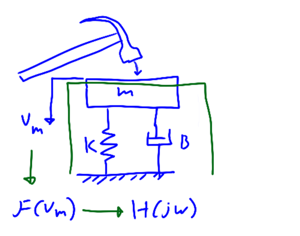
$$y(t) = h(t). \tag{8}$$

That is, h is the impulse response.

impulse response

9 A very interesting aspect of this result is that

$$H(j\omega) = \mathcal{F}(h). \tag{9}$$



That is, the Fourier transform of the impulse response is the frequency response function. A way to estimate, via measurement, the frequency response function (and transfer function) of a system is to input an impulse, measure and fit the response, then Fourier transform it. Of course, putting in an actual impulse and fitting the response, perfectly are impossible; however, estimates using approximations remain useful.

10 It is worth noting that frequency response/transfer function estimation is a significant topic of study, and many techniques exist. Another method is described in Lec. freq.sin.

System Identification

Example freq.fir−1

re: impulse response estimation of $H(j\omega)$

Estimate the frequency response function $H(j\omega)$ of a system from impulse response $h(t)$ "data". (We'll generate this data ourselves, simulating a measured impulse response.) We will not attempt to find the functional form of $H(j\omega)$, just its "numerical" form, i.e. we'll plot our estimate of the spectrum.

Note that if we wanted to find a functional estimate of $H(j\omega)$, it would behoove us to use Matlab's System Identification Toolbox.

Generate impulse response data

We need a system to simulate to get this (supposedly "measured") data. Let's define a transfer function

$$H(s) = \frac{s + 20}{s^2 + 4s + 20}. \tag{10}$$

```
sys = tf([1,20],[1,4,20])
```

```
sys =

      s + 20
  --------------
  s^2 + 4 s + 20

Continuous-time transfer function.
```

What are the poles?

```
poles = pole(sys)
```

```
poles =

  -2.0000 + 4.0000i
  -2.0000 - 4.0000i
```

This corresponds to a damped oscillator with natural frequency as follows.

```
abs(poles(1))
```

```
ans =

    4.4721
```

rad/s

Now let's find the impulse response.

$u_s(t)$ step

$\delta(t)$ impulse

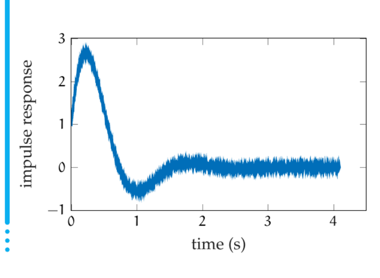
```
fs = 1000; % Hz .. sampling frequency
N = 2^12;
t_a = 0:1/fs:(N-1)/fs;
h_a = impulse(sys,t_a);
```

To make this seem a little more realistic as a "measurement," we should add some noise.

```
noise = 0.01*randn(N,1);
h_noisy = h_a + noise;
```

Plot the impulse response.

```
figure
plot(...
  t_a,h_noisy, ...
  'linewidth',1.5 ...
)
xlabel('time (s)')
ylabel('impulse response')
```



Discrete Fourier transform

The discrete Fourier transform will give us an estimate of the frequency spectrum of the system; that is, a numerical version of $H(j\omega)$.

```
H = fft(h_noisy);
```

Compute the one-sided magnitude spectrum.

```
H_mag = abs(H/fs); % note the scaling
H_mag = H_mag(1:N/2+1); % first half, only
```

Compute the one-sided phase spectrum.

```
H_pha = angle(H); % note the scaling
H_pha = H_pha(1:N/2+1); % first half, only
```

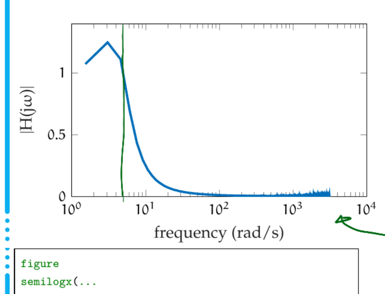
Now the corresponding frequencies.

```
f = fs*(0:(N/2))/N;
```

Plot the frequency response function

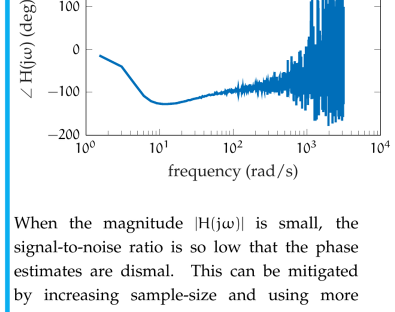
We like to use a logarithmic scale, at least in frequency, for the spectrum plots.

```
figure
semilogx(...
  2*pi*f,H_mag, ...
  'linewidth',1.5 ...
)
xlabel('frequency (rad/s)')
ylabel('|H(j\omega)|')
```



log scale

```
figure
semilogx(...
  2*pi*f,180/pi*H_pha, ...
  'linewidth',1.5 ...
)
xlabel('frequency (rad/s)')
ylabel('\angle H(j\omega) (deg)')
```



Bode Plots

When the magnitude $|H(j\omega)|$ is small, the signal-to-noise ratio is so low that the phase estimates are dismal. This can be mitigated by increasing sample-size and using more advanced techniques for estimating $H(j\omega)$, such as those available in Matlab's System Identification Toolbox.