freq.sin Sinusoidal input, frequency response

1 In this lecture, we explore the relationship—which turns out to be pretty chummy—between a system's frequency response function $H(j\omega)$ and its sinusoidal forced response.

2 Let's build from the frequency response function $H(j\omega)$ definition:

$$y(t) = \mathcal{F}^{-1} Y(\omega) \tag{1a}$$
$$= \mathcal{F}^{-1}(H(j\omega)U(\omega)). \tag{1b}$$

We take the input to be sinusoidal, with amplitude $A \in \mathbb{R}$, angular frequency ω_0, and phase ψ:

$$u(t) = A\cos(\omega_0 t + \psi). \tag{2}$$

The Fourier transform of the input, $U(\omega)$, can be constructed via transform identities from Table ft.1. This takes a little finagling. Let

$$p(t) = Aq(t), \tag{3a}$$
$$q(t) = r(t - t_0), \text{ and} \tag{3b}$$
$$r(t) = \cos\omega_0 t, \text{ where} \tag{3c}$$
$$t_0 = -\psi/\omega_0. \tag{3d}$$

The corresponding Fourier transforms, from Table ft.1, are

$$P(\omega) = AQ(\omega), \tag{4a}$$
$$Q(\omega) = e^{-j\omega t_0} R(\omega), \text{ and} \tag{4b}$$
$$R(\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0). \tag{4c}$$

Putting these together,

$$U(\omega) = A\pi\left(e^{j\psi\omega/\omega_0}\delta(\omega-\omega_0) + e^{j\psi\omega/\omega_0}\delta(\omega+\omega_0)\right)$$
$$= A\pi\left(e^{j\psi}\delta(\omega-\omega_0) + e^{-j\psi}\delta(\omega+\omega_0)\right)$$

(because δs)

3 And now we are ready to substitute into Eq. 1b; also applying the "linearity" property of the Fourier transform:

$$y(t) = A\pi\left(e^{j\psi}\mathcal{F}^{-1}(H(j\omega)\delta(\omega-\omega_0)) + e^{-j\psi}\mathcal{F}^{-1}(H(j\omega)\delta(\omega+\omega_0))\right). \tag{5}$$

The definition of the inverse Fourier transform gives

$$y(t) = \frac{A}{2}\left(e^{j\psi}\int_{-\infty}^{\infty} e^{j\omega t}H(j\omega)\delta(\omega-\omega_0)d\omega + \right.$$
$$\left. + e^{-j\psi}\int_{-\infty}^{\infty} e^{j\omega t}H(j\omega)\delta(\omega+\omega_0)d\omega\right). \tag{6}$$

Recognizing that δ is an even distribution ($\delta(t) = \delta(-t)$) and applying the sifting property of δ allows us to evaluate each integral:

$$y(t) = \frac{A}{2}\left(e^{j\psi}e^{j\omega_0 t}H(j\omega_0) + e^{-j\psi}e^{-j\omega_0 t}H(-j\omega_0)\right). \tag{7}$$

Writing H in polar form,

$$y(t) = \frac{A}{2}\left(e^{j(\omega_0 t+\psi)}|H(j\omega_0)|e^{j\angle H(j\omega_0)} + e^{-j(\omega_0 t+\psi)}|H(-j\omega_0)|e^{j\angle H(-j\omega_0)}\right) \tag{8}$$

The Fourier transform is conjugate symmetric—that is, $F(-\omega) = F^*(\omega)$—which allows us to further simply:

$|H(-j\omega)| = |H(j\omega)|$
$\angle H(-j\omega) = -\angle H(j\omega)$

$$y(t) = \frac{A|H(j\omega_0)|}{2}\left(e^{j(\omega_0 t+\psi)}e^{j\angle H(j\omega_0)} + e^{-j(\omega_0 t+\psi)}e^{-j\angle H(j\omega_0)}\right) \tag{9a}$$
$$= A|H(j\omega_0)|\frac{e^{j(\omega_0 t+\psi+\angle H(j\omega_0))} + e^{-j(\omega_0 t+\psi+\angle H(j\omega_0))}}{2}. \tag{9b}$$

Finally, Euler's formula yields something that deserves a box.

Equation 10 sinusoidal response in terms of $H(j\omega)$

For input $A\cos(\omega_0 t + \psi)$ to a system $H(j\omega)$ the forced response is $y(t) = A|H(j\omega_0)|\cos(\omega_0 t + \psi + \angle H(j\omega_0))$

4 This is a remarkable result. For an input sinusoid, a linear system has a forced response that

- ✓ is also a sinusoid,
- ✓ is at the same frequency as the input,
- ✓ differs only in amplitude and phase,
- ✓ differs in amplitude by a factor of $|H(j\omega)|$, and
- ✓ differs in phase by a shift of $\angle H(j\omega)$.

Now we see one of the key facets of the frequency response function: it governs how a sinusoid transforms through a system. And just think how powerful it will be once we combine it with the powerful principle of superposition and the mighty Fourier series representation of a function—as a "superposition" of sinusoids!