freq.per Periodic input, frequency response

1 Let a system H have a periodic input u represented by a Fourier series. For reals a_0, ω_1 (fundamental frequency), $\mathcal{A}_n$, and ϕ_n, let

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \mathcal{A}_n \sin(n\omega_1 t + \phi_n). \quad (1)$$

The nth harmonic is

which, from Equation 10 yields forced response

2 Applying the principle of superposition, the forced response of the system to periodic input u is

$$y(t) = \frac{a_0}{2} H(j0) + \sum_{n=1}^{\infty} \mathcal{A}_n |H(jn\omega_1)| \sin(n\omega_1 t + \phi_n + \angle H(jn\omega_1)). \quad (2)$$

3 Similarly, for inputs expressed as a complex Fourier series with components

$$u_n(t) = c_n e^{jn\omega_1 t}, \quad (3)$$

each of which has output

$$y_n(t) = c_n H(jn\omega_1) e^{jn\omega_1 t}, \quad (4)$$

the principle of superposition yields

$$y(t) = \sum_{n=-\infty}^{\infty} c_n H(jn\omega_1) e^{jn\omega_1 t}. \quad (5)$$

4 Eqs. 2 and 5 tell us that, for a periodic input, we obtain a periodic output with each harmonic ω_n amplitude scaled by $|H(j\omega_n)|$ and phase offset by $\angle H(j\omega_n)$. As a result, the response will usually undergo significant distortion, called phase distortion. The system H can be considered to filter the input by amplifying and suppressing different harmonics. This is why systems not intended to be used as such are still sometimes called "filters." This way of thinking about systems is very useful to the study of vibrations, acoustics, measurement, and electronics.

phase distortion
filter

5 All this can be visualized via a Bode plot, which is a significant aspect of its analytic power. An example of such a visualization is illustrated in Figure per.1.

Example freq.per−1 — re: filtering a square wave

In Example four.series-1, we found that a square wave of amplitude one has trigonometric Fourier series components

$$a_n = 0 \quad \text{and} \quad b_n = \frac{2}{n\pi}(1 - \cos(n\pi)) = \begin{cases} 0 & n \text{ even} \\ \frac{4}{n\pi} & n \text{ odd.} \end{cases}$$

Therefore, from the definitions of C_n and ϕ_n, with $b_n \geqslant 0$,

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\omega_n t) + b_n \sin(\omega_n t)$$

$$= \sum_{n=1}^{\infty} C_n \sin(\omega_n t + \phi_n)$$

$C_n = b_n$ and

$$\phi_n = \arctan \frac{b_n}{a_n} = \begin{cases} ¿ & \text{(indeterminate) for } n \text{ even} \\ \pi/2 & \text{for } n \text{ odd.} \end{cases}$$

$$C_n = \sqrt{a_n^2 + b_n^2} = \sqrt{0 + b_n^2} = b_n$$

Let this square wave be the input u to a second-order system with frequency response function $H(j\omega)$, natural frequency $\omega_N = \omega_5$ (fifth harmonic frequency), and damping ratio $\zeta = 0.1$.

Figure per.2 and Figure per.3 show the magnitude and phase spectra for input u, frequency response function $H(j\omega)$, and output y.

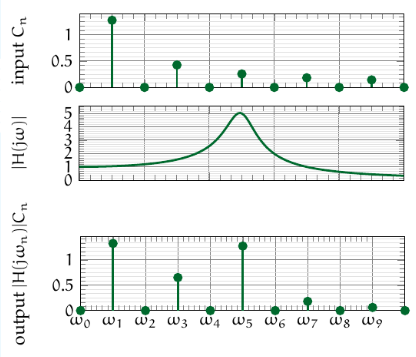

Figure per.2: the magnitude line spectrum C_n of the input, which is operated on by the measurement system with frequency response function $H(j\omega)$ to form the output magnitude line spectrum $|H(j\omega_n)|C_n$.

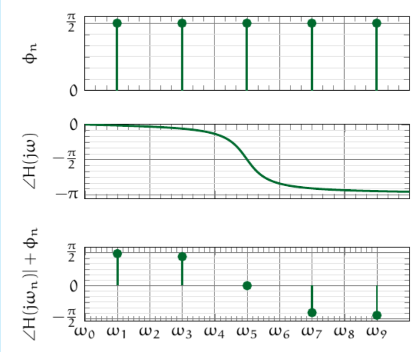

Figure per.3: the phase line spectrum ϕ_n of the input, which is operated on by the measurement system with frequency response function $H(j\omega)$ to form the output phase line spectrum $\angle H(j\omega_n) + \phi_n$.

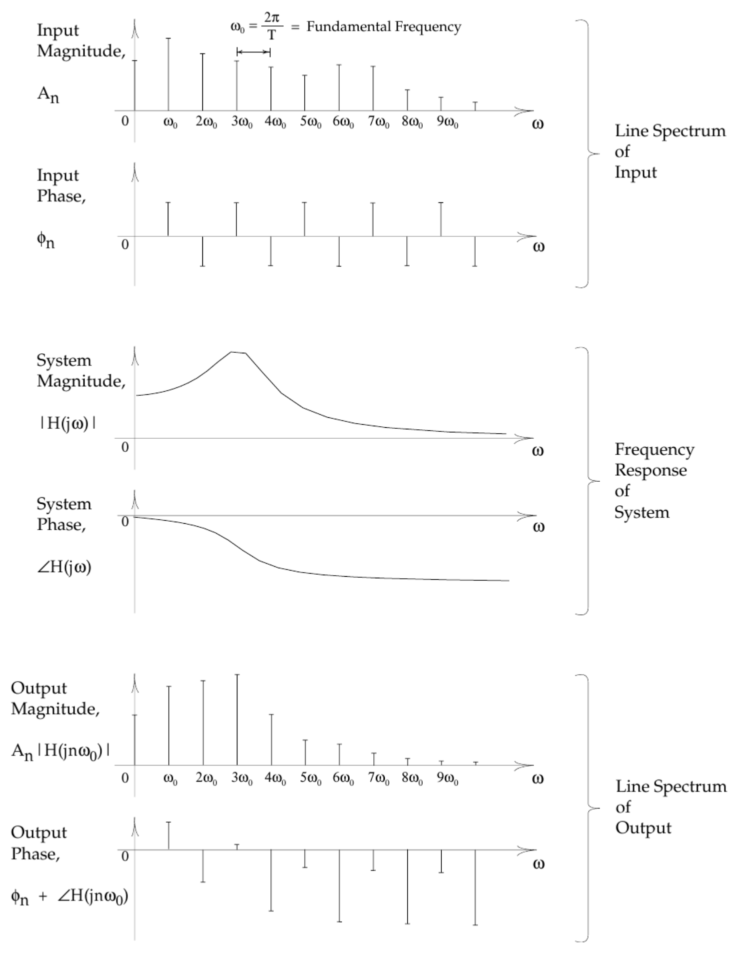

Figure per.1: response y of a system H to periodic input u.