lap.inv Inverse Laplace transforming

1 The inverse Laplace transform is a path integral in the s-plane, and it can be quite challenging to calculate. Therefore, software and tables such as Table ft.1 are typically applied, instead. In system dynamics, it is common to apply the inverse Laplace transform to a ratio (or products thereof) of polynomials in s like

$$\frac{a_m s^m + a_{m-1} s^{m-1} + \cdots + a_0}{b_n s^n + b_{n-1} s^{n-1} + \cdots + b_0} \tag{1}$$

for $a_i, b_i \in \mathbb{R}$. However, inverse transforms of general ratios such as these do not appear in the tables. Instead, low-order polynomial ratios do appear and have simple inverse Laplace transforms. Suppose we could decompose Eq. 1 into smaller additive terms. Due to the linearity property of the inverse Laplace transform, each transform could be calculated separately and consequently summed.

2 The name given to the process of decomposing Eq. 1 into smaller additive terms is called partial fraction expansion[9]. It is not particularly difficult, but it is rather tedious. Fortunately, several software tools have been developed for this expansion.

partial fraction expansion

9. Rowell and Wormley, System Dynamics: An Introduction, App. C.

Inverse transform with a partial fraction expansion in Matlab

3 Matlab's Symbolic Math toolbox function `partfrac` is quite convenient.

```
help partfrac
```

4 Let's apply this to an example.

Example lap.inv−1

What is the inverse Fourier transform image of

$$F(s) = \frac{s^2+2s+2}{s^2+6s+36} \cdot \frac{6}{s+6}? \tag{2}$$

First, define a symbolic s.

```
syms s 'complex'
```

Now we can define F, a symbolic expression for $F(s)$.

```
F = (s^2 + 2*s + 2)/(s^2 + 6*s + 36)*6/(s+6);
```

Now all that remains is the apply `partfrac`.

```
F_pf = partfrac(F)
```

```
F_pf =
13/(3*(s + 6)) + ((5*s)/3 - 24)/(s^2 + 6*s + 36)
```

$$= \frac{13}{3(s+6)} \checkmark + \frac{\frac{5}{3}s - 24}{s^2+6s+36}$$

Now consider the Laplace transform table. The first term can easily be inverted:

$$\begin{aligned}\mathcal{L}^{-1}\left(\frac{13}{3}\cdot\frac{1}{s+6}\right) &= \frac{13}{3}\mathcal{L}^{-1}\frac{1}{s+6} && \text{(linearity)}\\ &\frac{13}{3}e^{-6t}. && \text{(table)}\end{aligned}$$

The second term, call it F_2, is not quite as obvious, but the preimage

$$\frac{s-a}{(s-a)^2+\omega^2} \tag{3}$$

is close. Let's first make the numerator match:

$$\frac{5}{3}\,\frac{s-\frac{72}{5}}{s^2+6s+36} \qquad a = \frac{72}{5}$$

$$\frac{5}{3}s - 24 = \frac{5}{3}\left(s - \frac{72}{5}\right), \tag{4}$$

so $a_1 = 72/5$. Now we need the term $(s-a_1)^2$ in the denominator. Asserting the equality

$$\left(s-\frac{72}{5}\right)^2 + \omega^2$$

$$\begin{aligned} s^2+6s+36 &= (s-a_2)^2+\omega^2\\ &= s^2 - 2a_2 s + a_2^2 + \omega^2.\end{aligned}$$

$$= s^2 - 2\,\frac{72}{5}s + \left(\frac{72}{5}\right)^2 + \omega^2$$

$$2\,\frac{72}{5} = 28.8 \neq 6$$

Equating the s^0 coefficents yields $\omega^2 = 36 - a_2^2$ and equating the s coefficient yields $a_2 = -3 \neq a_1 = 72/5$, so no cigar! What if we "force" the rule by using a new $a_1' = a_2$, which can be achieved by adding a term (and subtracting it elsewhere)? We need $a_1' = -3$, so if we add (and subtract) a term

$$\frac{\frac{5}{3}(a_1 - a_1')}{(s-a_2)^2+\omega^2},$$

like

$$F_2 = \frac{\frac{5}{3}(s-a_1)}{(s-a_2)^2+\omega^2} + \frac{\frac{5}{3}(a_1-a_1')}{(s-a_2)^2+\omega^2} - \frac{\frac{5}{3}(a_1-a_1')}{(s-a_2)^2+\omega^2}$$

we can combine the first two terms to yield

$$F_2 = \frac{\frac{5}{3}(s-a_1')}{(s-a_2)^2+\omega^2} - \frac{\frac{5}{3}(a_1-a_1')}{(s-a_2)^2+\omega^2}$$

where we recall that $a_1' = a_2$ by construction. Now the expression is

$$F_2 = \frac{\frac{5}{3}(s-a_2)}{(s-a_2)^2+\omega^2} - \frac{\frac{5}{3}(a_1-a_2)}{(s-a_2)^2+\omega^2}$$

The first term is, by construction, in the Laplace transform table. The second term is close to

$$\frac{\omega}{(s-a)^2+\omega^2}$$

for which we must make the numerator equal ω. Our $\omega^2 = 36 - a_2^2 = 27$, so $\omega = \pm\sqrt{27}$. The current numerator is

$$\begin{aligned}\frac{5}{3}(a_1-a_2) &= \frac{5}{3}\left(\frac{72}{5}+3\right)\\ &= 29.\end{aligned}$$

So we factor out $29/\sqrt{27}$ to yield

$$\frac{\frac{29}{\sqrt{27}}\omega}{(s-a_2)^2+\omega^2}$$

Returning to F_2, we have arrived at

$$F_2 = \frac{\frac{5}{3}(s-a_2)}{(s-a_2)^2+\omega^2} - \frac{\frac{29}{\sqrt{27}}\omega}{(s-a_2)^2+\omega^2}$$

Now the inverse transform is

$$\begin{aligned}\mathcal{L}^{-1}F_2 &= \frac{5}{3}\mathcal{L}^{-1}\frac{(s-a_2)}{(s-a_2)^2+\omega^2} - \frac{29}{\sqrt{27}}\mathcal{L}^{-1}\frac{\omega}{(s-a_2)^2+\omega^2} && \text{(linearity)}\\ &= \frac{5}{3}e^{a_2 t}\cos\omega t - \frac{29}{\sqrt{27}}e^{a_2 t}\sin\omega t.\end{aligned}$$

Simple! Putting it all together, then,

$$F(s) = \frac{13}{3}e^{-6t} + \frac{5}{3}e^{-3t}\cos(3\sqrt{3}t) - \frac{29}{3\sqrt{3}}e^{-3t}\sin(3\sqrt{3}t).$$

5 You may have noticed that even with Matlab's help with the partial fraction expansion, the inverse Laplace transform was a bit messy. This will motivate you to learn the technique in the next section.

Just clubbing it with Matlab

6 Sometimes we can just use Matlab (or a similar piece of software) to compute the transform.

7 Matlab's Symbolic Math toolbox function for the inverse Laplace transform is `ilaplace` (and for the Laplace transform, `laplace`).

```
help ilaplace
```

8 Let's apply this to the same example.

Example lap.inv−2

What is the inverse Fourier transform image of

$$F(s) = \frac{s^2+2s+2}{s^2+6s+36} \cdot \frac{6}{s+6}? \tag{5}$$

Use Matlab's `ilaplace`.

First, define a symbolic s.

```
syms s 'complex'
```

Now we can define F, a symbolic expression for $F(s)$.

```
F = (s^2 + 2*s + 2)/(s^2 + 6*s + 36)*6/(s+6);
```

Now all that remains is the apply `ilaplace`.

```
F_pf = ilaplace(F)
```

```
F_pf =
(13*exp(-6*t))/3 + (5*exp(-3*t)*(cos(3*3^(1/2)*t) -
↪  (29*3^(1/2)*sin(3*3^(1/2)*t))/15))/3
```

This is easily seen to be equivalent to our previous result

$$F(s) = \frac{13}{3}e^{-6t} + \frac{5}{3}e^{-3t}\cos(3\sqrt{3}t) - \frac{29}{3\sqrt{3}}e^{-3t}\sin(3\sqrt{3}t).$$