lap.pr Properties of the Laplace transform

1 The Laplace transform has several important properties, several of which follow from the simple fact of its integral definition. We state the properties without proof, but several are easy to show and make good exercises.

Existence

2 As we have already seen, the Laplace transform exists for more functions than does the Fourier transform. Let $f : \mathbb{R}_+ \to \mathbb{R}$ have a finite number of finite-magnitude discontinuites. If there can be found $M, \alpha \in \mathbb{R}$ such that

$$|f(t)| \leqslant Me^{\alpha t} \quad \forall t \in \mathbb{R}_+ \tag{1}$$

then the transform exists (converges) for $\sigma > \alpha$.

3 Note that this is a sufficient condition, not necessary. That is, there may be (and are) functions for which a transform exists that do not meet the condition above.

$10e^{27t} \le Me^{\alpha t}$ ✓

$2^t \le Me^{\alpha t}$ ✓ $M=1$ $\alpha=1$

$e^{t^2} \le Me^{\alpha t}$ ✗

Linearity

4 The Laplace transform is a linear map. Let $a, b \in \mathbb{R}$; $f, g \in T$ where T is a set of functions of nonnegative time t; and F, G the Laplace transform images of f, g. The following identity holds:

$$\mathcal{L}(af(t) + bg(t)) = aF(s) + bG(s). \tag{2}$$

Time-shifting

5 Shifting the time-domain function $f(t)$ in time corresponds to a simple product in the s-domain Laplace transform image. Let the Laplace transform image of $f(t)$ be $F(s)$ and $\tau \in \mathbb{R}$. The following identity holds:

$$\mathcal{L}(f(t + \tau)) = e^{s\tau}F(s). \tag{3}$$

Time-differentiation

6 Differentiating the time-domain function $f(t)$ with respect to time yields a simple relation in the s-domain. Let $F(s)$ be the Laplace transform image of $f(t)$ and $f(0)$ the value of f at $t = 0$. The following identity holds:[7]

$$\mathcal{L}\frac{df}{dt} = sF(s) - f(0). \tag{4}$$

7. For this reason, it is common for s to be called the differentiator, but this is imprecise and pretty bush league.

$H(s) = s$ $\quad H(s) = \dfrac{Y(s)}{U(s)}$

$s = \dfrac{Y(s)}{U(s)}$ $\quad sU(s) = Y(s)$

$\dfrac{du}{dt} = y(t)$

Time-integration

7 Similarly, integrating the time-domain function $f(t)$ with respect to time yields a simple relation in the s-domain. Let $F(s)$ be the Laplace transform image of $f(t)$. The following identity holds:[8]

$$\mathcal{L}\int_0^t f(\tau)\, d\tau = \frac{1}{s}F(s). \tag{5}$$

8. For this reason, it is common for $1/s$ to be called the integrator.

$H(s) = \dfrac{1}{s}$

$\dfrac{1}{s} = \dfrac{Y(s)}{U(s)}$ $\quad \dfrac{1}{s}U(s) = Y(s)$

$\int_0^t u(\tau)\,d\tau = y(t)$

Convolution

8 The convolution operator $*$ is defined for real functions of time f, g by

$$(f * g)(t) \equiv \int_{-\infty}^{\infty} f(\tau)g(t - \tau)\, d\tau. \tag{6}$$

This too has a simple Laplace transform. Let F, G be the Laplace transforms of f, g. The following identity holds:

$$\mathcal{L}(f * g)(t) = F(s)G(s). \tag{7}$$

Final value theorem

final value theorem

9 The final value theorem is a property of the Laplace transform. This theorem allows the computation of constant time-domain steady-state values from the frequency domain, which can be quite convenient when the inverse Laplace transform is elusive. Let $f(t)$ have tranform $F(s)$ and its time-derivative have an existing transform. If the limit in time exists,

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s). \tag{8}$$

Note that if the steady-state of $f(t)$ is not a constant (e.g. it is sinusoidal), the limit does not exist.