

lap.sol Solving io ODEs with Laplace

- Laplace transforms provide a convenient method for solving input-output (io) ordinary differential equations (ODEs).
- Consider a dynamic system described by the **IO ODE** —with t time, y the output, u the input, constant coefficients a_i, b_j , order n , and $m \leq n$ for $n \in \mathbb{N}_0$ —as:

$$\begin{aligned} \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_1 \frac{dy}{dt} + a_0 y &= \\ b_m \frac{d^m u}{dt^m} + b_{m-1} \frac{d^{m-1} u}{dt^{m-1}} + \dots + b_1 \frac{du}{dt} + b_0 u. \end{aligned} \quad (1)$$

Re-written in summation form,

$$\sum_{i=0}^n a_i y^{(i)}(t) = \sum_{j=0}^m b_j u^{(j)}(t), \quad (2)$$

where we use Lagrange's notation for derivatives, and where, **by convention**, $a_n = 1$.

- The Laplace transform \mathcal{L} of Eq. 2 yields

$$\mathcal{L} \sum_{i=0}^n a_i y^{(i)}(t) = \mathcal{L} \sum_{j=0}^m b_j u^{(j)}(t) \Rightarrow \quad (3a)$$

$$\sum_{i=0}^n a_i \mathcal{L}(y^{(i)}(t)) = \sum_{j=0}^m b_j \mathcal{L}(u^{(j)}(t)). \quad (\text{linearity})$$

In the next move, we recursively apply the **differentiation** property to yield the following

$$\sum_{i=0}^n a_i \left(s^i Y(s) + \underbrace{\sum_{k=1}^i s^{i-k} y^{(k)}(0)}_{I_1(s)} \right) = \sum_{j=0}^m b_j s^j U(s), \quad (4)$$

where terms in $I_1(s)$ arise from the **initial conditions**. Splitting the left outer sum and

solving for $Y(s)$,

$$\sum_{i=0}^n a_i s^i Y(s) + \sum_{i=0}^n a_i I_1(s) = \sum_{j=0}^m b_j s^j U(s) \Rightarrow \quad (5a)$$

$$\sum_{i=0}^n a_i s^i Y(s) = \sum_{j=0}^m b_j s^j U(s) - \sum_{i=0}^n a_i I_1(s) \Rightarrow \quad (5b)$$

$$Y(s) \sum_{i=0}^n a_i s^i = U(s) \sum_{j=0}^m b_j s^j - \sum_{i=0}^n a_i I_1(s) \Rightarrow \quad (5c)$$

$$Y(s) = \frac{\sum_{j=0}^m b_j s^j U(s)}{\sum_{i=0}^n a_i s^i} - \frac{\sum_{i=0}^n a_i I_1(s)}{\sum_{i=0}^n a_i s^i}. \quad (5d)$$

- So we have derived the **Laplace transform image** $Y(s)$ in terms of the forced and free responses (still in the s -domain, of course)! For a solution in the time-domain, we must inverse Laplace transform:

$$y(t) = \underbrace{\mathcal{L}^{-1} Y_{in}(s)}_{y_{in}(t)} + \underbrace{\mathcal{L}^{-1} Y_{out}(s)}_{y_{out}(t)}. \quad (6)$$

This is an important result!

Example lap.sol-1

Consider a system with step input $u(t) = 7u_s(t)$, output $y(t)$, and io ODE

$$\ddot{y} + 2\dot{y} + y = 2u. \quad (7)$$

Solve for the forced response $y_{in}(t)$ with Laplace transforms.

- From Eq. 6,

$$y_{in}(t) = \mathcal{L}^{-1} Y_{in}(s)(t)$$

$$= \mathcal{L}^{-1} \left(\frac{\sum_{j=0}^m b_j s^j U(s)}{\sum_{i=0}^n a_i s^i} \right) \quad \text{Eq. 5d}$$

$$= \mathcal{L}^{-1} \left(\frac{2}{s^2 + 2s + 1} U(s) \right). \quad \text{Eq. 7}$$

We can **Laplace transform** $u(t)$ for $U(s)$:

$$\begin{aligned} U(s) &= (\mathcal{L}u)(s) \\ &= 7(\mathcal{L}u_s)(s) \\ &= \frac{7}{s}, \end{aligned}$$

where the last equality follows from a transform easily found in Table lap.1.

- Returning to the time response **with**

$$\begin{aligned} y_{in}(t) &= \mathcal{L}^{-1} \left(\frac{2}{s^2 + 2s + 1} U(s) \right) \\ &= \mathcal{L}^{-1} \left(\frac{2}{s^2 + 2s + 1} \cdot \frac{7}{s} \right). \end{aligned}$$

- We can use Matlab's Symbolic Math toolbox function `partfrac` to perform the partial fraction expansion.

```
syms s 'complex'
Y = 2/(s^2 + 2*s + 1)*7/s;
Y_pf = partfrac(Y)
```

```
Y_pf =
14/s - 14/(s + 1)^2 - 14/(s + 1)
```

Or, a little nicer to look at,

$$Y(s) = 14 \left(\frac{1}{s} - \frac{1}{(s+1)^2} - \frac{1}{s+1} \right).$$

Substituting this into our solution,

$$\begin{aligned} y_{in}(t) &= 14 \mathcal{L}^{-1} \left(\frac{1}{s} - \frac{1}{(s+1)^2} - \frac{1}{s+1} \right) \\ &= 14 \left(\mathcal{L}^{-1} \frac{1}{s} - \mathcal{L}^{-1} \frac{1}{(s+1)^2} - \mathcal{L}^{-1} \frac{1}{s+1} \right) \\ &= 14 (u_s(t) - te^{-t} - e^{-t}) \quad (\text{Table lap.1}) \\ &= 14 (u_s(t) - (t+1)e^{-t}). \end{aligned}$$

So the forced response starts at 0 and decays **pseudo-exponentially** to a steady 14.

$$y_{in}(\infty) = \lim_{s \rightarrow 0} s \frac{2}{s^2 + 2s + 1} \frac{7}{s} = \frac{2 \cdot 7}{1} = 14 \checkmark$$