nlin.lin Linearization

1 A common method for dealing with a nonlinear system is to linearize it: transform it such that its state equation is linear. A linearized model is typically only valid in some neighborhood of state-space. This neighborhood is selected by choosing an operating point $\mathbf{x}_o$ used in the linearization process. We use two considerations when choosing an operating point:

linearize

operating point

1. that implied by the name—it should be in a region of state-space in which the state will stay throughout the system's operation—and
2. the validity of the model near the operating point.

Due to the fact that nonlinear systems tend to be more-linear near equilibria, the second consideration frequently suggests we choose one as an operating point: $\mathbf{x}_o = \overline{\mathbf{x}}$.

Taylor series expansion

2 A Taylor series expansion of Eq. 1a about an operating point $\mathbf{x}_o, \mathbf{u}_o$ (for a nonautonomous system) yields polynomial terms that are linear, quadratic, etc. in $\mathbf{x}$ and $\mathbf{u}$. If we keep only the linear terms and define new state and input variables

$$\mathbf{x}^* = \mathbf{x} - \mathbf{x}_o \quad \text{and} \quad \mathbf{u}^* = \mathbf{u} - \mathbf{u}_o \tag{1a}$$

we get a linear state equation

$$\frac{d\mathbf{x}^*}{dt} = A\mathbf{x}^* + B\mathbf{u}^* \tag{1b}$$

where the matrix components are given by

$$A_{ij} = \left.\frac{\partial f_i}{\partial x_j}\right|_{\mathbf{x}_o,\mathbf{u}_o} \quad \text{and} \quad B_{ij} = \left.\frac{\partial f_i}{\partial u_j}\right|_{\mathbf{x}_o,\mathbf{u}_o}. \tag{1c}$$

These first-derivative matrices are generally called Jacobian matrices.

Jacobian

3 This result also applies to autonomous equations if we drop the $B\mathbf{u}^*$ term.

Example nlin.lin−1

re: hardening spring

Consider a vehicle suspension system that is overloaded such that its springs are exhibiting hardening behavior such that a lumped-parameter constitutive equation for the springs (collectively) is

$$f_k = kx_k + ax_k^3 \tag{2}$$

where f_k is the force, x_k the displacement, and $k, a > 0$ constant parameters of the spring.

a. Develop a (nonlinear) spring-mass-damper linear graph model for the vehicle suspension with input position source X_s.

b. Derive a nonlinear state-space model from the linear graph model using the state vector

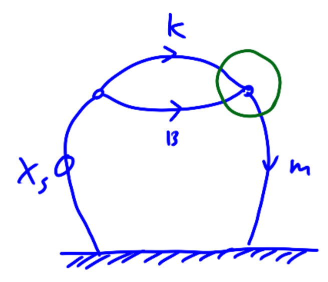
$$\mathbf{x} = \begin{bmatrix} x_m & v_m \end{bmatrix}^\top. \tag{3}$$

c. Linearize the system about the operating point

$$\mathbf{x}_o = \begin{bmatrix} 1 & 0 \end{bmatrix}^\top \quad \text{and} \quad \mathbf{u}_o = \begin{bmatrix} 0 \end{bmatrix} \tag{4}$$

by computing the A, B, and E matrices of the linearized system.[a]

a. The E matrix is the Jacobian with respect to the time-derivative of the input: $\dot{\mathbf{u}}$, which arises occasionally.



$\dot{V}_m = \frac{1}{m} f_m$ $\quad f_m = f_B + f_K$

$f_B = BV_B$ $\quad V_s = V_B + V_m = \dot{X}_s \Rightarrow V_B = \dot{X}_s - V_m$

$f_K = Kx_K + ax_K^3$ $\quad X_s = X_K + X_m \Rightarrow X_K = X_s - X_m$

$$\begin{bmatrix} \dot{x}_m \\ \dot{v}_m \end{bmatrix} = f\left(\begin{bmatrix} x_m \\ v_m \end{bmatrix}, \begin{bmatrix} X_s \end{bmatrix} \right)$$

$$\dot{x}_m = v_m$$

$$\dot{v}_m = \frac{1}{m} f_m = \frac{1}{m}(f_B + f_K)$$
$$= \frac{1}{m}(Bv_B + kx_k + ax_k^3)$$
$$= \frac{1}{m}(B(\dot{X}_s - v_m) + k(X_s - x_m) + a(x_s - x_m)^3)$$

$$f\left(\begin{bmatrix} x_m \\ v_m \end{bmatrix}, \begin{bmatrix} X_s \end{bmatrix} \right) = \begin{bmatrix} v_m \\ \frac{1}{m}(B(\dot{x}_s - v_m) + k(x_s - x_m) + a(x_s - x_m)^3) \end{bmatrix}$$

$A_{11} = 0 \qquad A_{12} = 1$

$$A_{21} = \left.\frac{-k + 3a(x_s - x_m)^2(-1)}{m}\right|_{x_o, u_o} = \frac{-k - 3a(0-1)^2}{m} = \frac{-k-3a}{m}$$

$A_{22} = -B/m$

$$A = \begin{bmatrix} 0 & 1 \\ \frac{-k-3a}{m} & -B/m \end{bmatrix}$$

$$B_{11} = 0 \qquad B_{21} = \left.\frac{k + 3a(x_s - x_m)^2}{m}\right|_{x_o, u_o} = \frac{k + 3a(0-1)^2}{m} = \frac{k+3a}{m}$$

$$B = \begin{bmatrix} 0 \\ \frac{k+3a}{m} \end{bmatrix}$$

$$\dot{x} = Ax + Bu + E\dot{u} \qquad E = \begin{bmatrix} 0 \\ B/m \end{bmatrix}$$