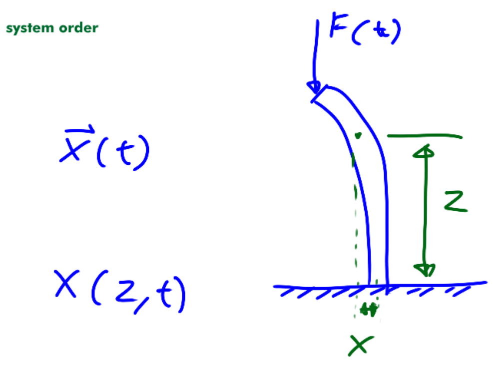
nlin.char Nonlinear system characteristics

1 Characterizing nonlinear systems can be challenging without the tools developed for linear system characterization. However, there are ways of characterizing nonlinear systems, and we'll here explore a few.

Those in-common with linear systems

2 As with linear systems, the system order is either the number of state-variables required to describe the system or, equivalently, the highest-order derivative in a single scalar differential equation describing the system.

system order



3 Similarly, nonlinear systems can have state variables that depend on time alone or those that also depend on space (or some other independent variable). The former lead to ordinary differential equations (ODEs) and the latter to partial differential equations (PDEs).

4 Equilibrium was already considered in Chapter nlin.

Stability

5 In terms of system performance, perhaps no other criterion is as important as stability.

Definition nlin.1: Stability

If x is perturbed from an equilibrium state $\bar{x}$, the response $x(t)$ can:

1. asymptotically return to $\bar{x}$ (asymptotically stable),
2. diverge from $\bar{x}$ (unstable), or
3. remain perturned or oscillate about $\bar{x}$ with a constant amplitude (marginally stable).

Notice that this definition is actually local: stability in the neighborhood of one equilibrium may not be the same as in the neighborhood of another.

6 Other than nonlinear systems' lack of linear systems' eigenvalues, poles, and roots of the characteristic equation from which to compute it, the primary difference between the stability of linear and nonlinear systems is that nonlinear system stability is often difficult to establish globally. Using a linear system's eigenvalues, it is straightforward to establish stable, unstable, and marginally stable subspaces of state-space (via transforming to an eigenvector basis). For nonlinear systems, no such method exists. However, we are not without tools to explore nonlinear system stability. One mathematical tool to consider is Lyapunov stability theory, which is beyond the scope of this course, but has good treatments in[2] and[3].

Lyapunov stability theory

2. William L Brogan. Modern Control Theory. Third. Prentice Hall, 1991, Ch. 10.

3. A. Choukchou-Braham andothers. Analysis and Control of Underactuated Mechanical Systems. SpringerLink : B ucher. Springer International Publishing, 2013. ISBN: 9783319026367, App. A.

Qualities of equilibria

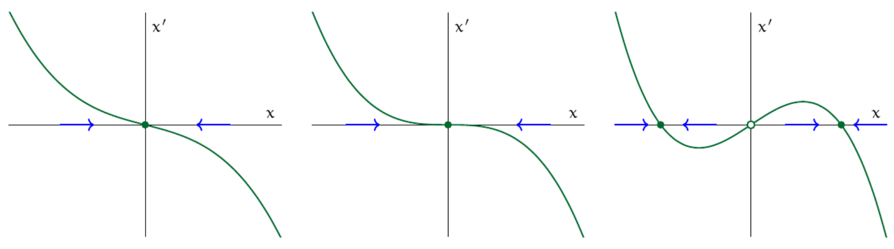
7 Equilibria (i.e. stationary points) come in a variety of qualities. It is instructive to consider the first-order differential equation in state variable x with real constant r:

$$x' = rx - x^3. \quad (1)$$

If we plot x' versus x for different values of r, we obtain the plots of Fig. char.1.

8 By definition, equilibria occur when $x' = 0$, so the x-axis crossings of Fig. char.1 are equilibria. The blue arrows on the x-axis show the direction of state change x', quantified by the plots. For both (a) and (b), only one equilibrium exists: $x = 0$. Note that the blue arrows in both plots point toward the equilibrium. In such cases—that is, when a neighborhood exists around an equilibrium for which state changes point toward the equilibrium—the equilibrium is called an attractor or sink. Note that attractors are stable.

attractor
sink
stable



(a) $r < 0$ (b) $r = 0$ (c) $r > 0$

Figure char.1: plots of x' versus x for Eq. 1.

9 Now consider (c) of Fig. char.1. When $r > 0$, three equilibria emerge. This change of the number of equilibria with the changing of a parameter is called a bifurcation. A plot of bifurcations versus the parameter is called a bifurcation diagram. The $x = 0$ equilibrium now has arrows that point away from it. Such an equilibrium is called a repeller or source and is unstable. The other two equilibria here are (stable) attractors. Consider a very small initial condition $x(0) = \epsilon$. If $\epsilon > 0$, the repeller pushes away x and the positive attractor pulls x to itself. Conversely, if $\epsilon < 0$, the repeller again pushes away x and the negative attractor pulls x to itself.

bifurcation

bifurcation diagram

repeller
source
unstable

10 Another type of equilibrium is called the saddle: one which acts as an attractor along some lines and as a repeller along others. We will see this type in the following example.

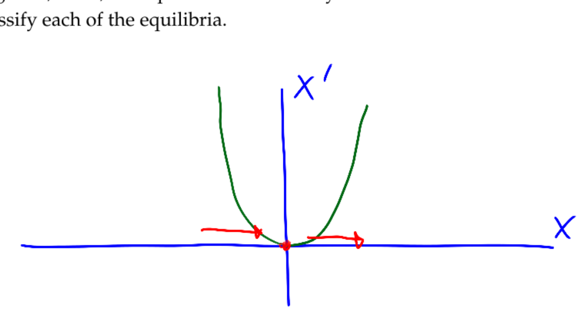
saddle

Example nlin.char-1 **re: Saddle bifurcation**

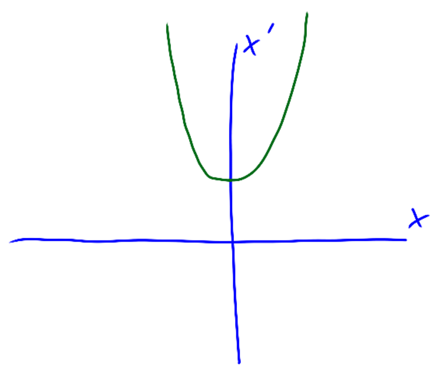
Consider the dynamical equation

$$x' = x^2 + r \quad (2)$$

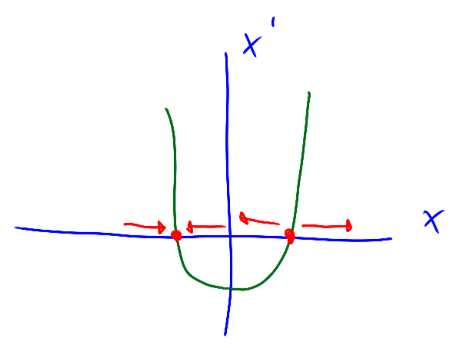
with r a real constant. Sketch x' vs x for negative, zero, and positive r. Identify and classify each of the equilibria.



r=0
Saddle



r>0
no stationary points



r<0

stationary points x<0 stable
x>0 unstable