stab.tf Stability from the transfer function

Stability from the poles of a closed-loop transfer function

From our definitions in terms of the free response (Lecture stab.intro), we see that a closed-loop LTI system is assymptotically stable if all its poles[5] have negative real parts (i.e. are in the left half-plane).
Conversely, a closed-loop LTI system is unstable if it has at least one pole with a positive real part (i.e. in the right half-plane) and/or has poles of multiplicity greater than one on the imaginary axis.
Finally, a closed-loop LTI system is marginally stable if it is not unstable but has at least one pole with zero real part (i.e. on the imaginary axis) and if none of these has multiplicity greater than one.

5. Recall that poles, eigenvalues, and roots of the characteristic equation are all equivalent.

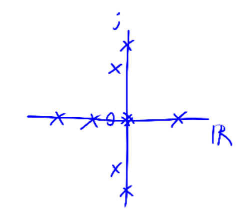
$\frac{1}{s-1} \rightarrow \frac{dy}{dt} - y = u \rightarrow \lambda - 1 = 0$

$\lambda = 1$

$y(t) = Ce^{t}$

$\frac{s+1}{(s+2)(s+3)}$

Poles $-2, -3$



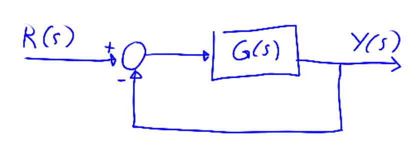
Example stab.tf−1

re: Stability of a closed−loop transfer function from its poles

Given the plant transfer function

$$G(s) = \frac{1}{(s^2+3)}$$

find the unity (negative) feedback closed-loop transfer function and comment on its stability. Let the command be $R(s)$ and the output $Y(s)$.



$\frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)}$

$= \frac{\frac{1}{s^2+3}}{1+\frac{1}{s^2+3}} \quad \frac{s^2+3}{s^2+3}$

$= \frac{1}{s^2+3+1} = \frac{1}{s^2+4}$

$s^2 + 4 = 0$

$s^2 = -4$

$s = \pm\sqrt{-4}$

$= \pm j2$

marginally stable

Stability from the form of a closed-loop transfer function

Let $a_i, b_i, c \in \mathbb{R}$ be constant coefficients and the denominator of a closed-loop transfer function be the polynomial

$$b_n s^n + b_{n-1}s^{n-1} + \cdots + b_0 = c(s-a_1)(s-a_2)\cdots(s-a_n). \quad (1)$$

If a system is stable, it must have all left half-plane poles, so

$s - a_i = 0$

$s = a_i$

1. all a_i must have negative real parts, which (non-obviously) implies that
2. all b_i must be positive and, additionally,
3. all b_i must be nonzero for $0 \leqslant i \leqslant n$ (i.e. no "missing" powers of s).

However, these b_i conditions are merely necessary conditions for stability, meaning that they are necessary for stability, but not sufficient (something more is needed to ensure stability).[6] However, if they are not met, this is a sufficient condition to draw the conclusion that the control system is unstable (i.e. nothing more needed).

6. The logical statement $P \Rightarrow Q$ means P is sufficient for Q and Q is necessary for P. That is, if P then Q (sufficiency) and if not Q then not P (necessity). Necessity and sufficiency are duals. Let P be "the system is stable" and Q be "all b_i are positive." Then if any b_i is negative ($\neg Q$), then the system is unstable ($\neg P$). But if Q, it does not necessarily follow that P—more information is required.

Example stab.tf−2

re: Stability of a closed−loop transfer function by inspection

Given the closed-loop transfer functions

$$G_1(s) = \frac{s+4}{(s+3)(s+10)(s+22)}, \quad G_2(s) = \frac{s^2+2s+5}{s^2-5s+8},$$

$$G_3(s) = \frac{1}{s^3+s+4}, \text{ and } \quad G_4(s) = \frac{s^2+5}{s^4+3s^3+s^2+s+3}.$$

comment on the stability of each without solving for poles.

$s + a = 0 \Rightarrow s = -a$ $\quad G_1$ stable

$s^2 - 5s + 8$ $\quad G_2$ unstable

$s^3 + 0s^2 + s + 4$ $\quad G_3$ unstable

$s^4 + 3s^3 + s^2 + s + 3$ $\quad G_4$ may be stable