A.01 Complex functions

A complex function f maps a subset of the complex plane to the complex plane (i.e. $f : \mathbb{C} \to \mathbb{C}$). For instance, a complex function f can map a single complex number s_0 to another $s_1 = f(s_0)$.

A curve in the complex plane is defined as a continuous function mapping a closed interval of the reals to the complex plane. A contour is defined as a directed curve consisting of a finite set of directed smooth curves, the final endpoint of which is identical to the starting point (Fig. A.01.1 shows a plot of a contour Γ).

A contour can be mapped by a complex function, and this is our primary concern. The image of a contour Γ mapped by a complex function f is itself a contour $f(\Gamma)$, as shown in Fig. A.01.2.

Complex functions are of interest in control theory because transfer functions, one of the central mathematical objects of control theory, are complex functions. The utility of evaluating and mapping contours with complex functions arises especially in root-locus design and frequency response design (especially for the Nyquist stability criterion).

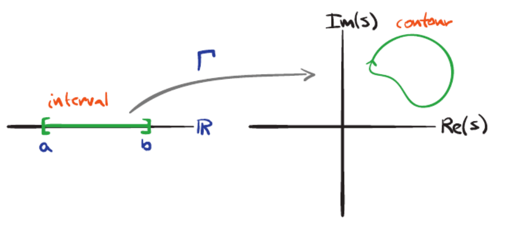

Figure A.01.1: illustrating the definition of a complex contour Γ.

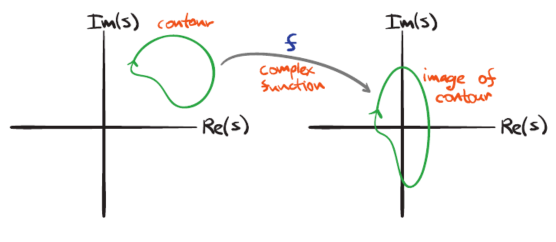

Figure A.01.2: a representation of a complex function f mapping a contour.

Example A.01–1 **re: transfer function mapping a single point**

Map the complex point $s = 1 + j3$ with the transfer (complex) function

$$H(s) = \frac{s+4}{s-1}.$$

Sometimes we say that we are "evaluating" the transfer function at the point $s = 1 + j3$.

$$H(1+j3) = \frac{(1+j3)+4}{(1+j3)-1} = \frac{5+j3}{j3}\frac{j}{j} = \frac{5j-3}{-3} = \frac{-5j}{3} + 1 = 1 - \frac{5}{3}j$$

A geometric interpretation of complex functions

It is often helpful to interpret the complex mapping of a point or a contour geometrically. Let us consider a transfer (complex) function $H(s)$ with complex zeros z_i, complex poles p_j, and real scaling factor k. Considering each factored term of the transfer function in terms of its magnitude and phase, we can write the magnitude and phase of the transfer function as follows.

Equation 1 magnitude and phase of a transfer function

$$|H(s)| = |K| \frac{\prod_i |s - z_i|}{\prod_j |s - p_j|}$$

$$\angle H(s) = \sum_i \angle(s - z_i) - \sum_j \angle(s - p_j)$$

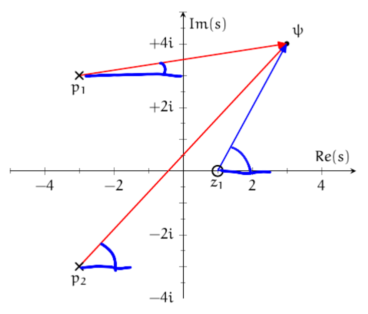

Figure A.01.3: an example of a geometric interpretation of the evaluation of a complex function with poles $p_{1,2}$ and zero z_1 at a complex value $s = \psi$.

We can interpret this geometrically as follows. Let us consider the evaluation of Eq. 1 at a specific complex value ψ. The differences $\psi - z_i$ and $\psi - p_i$ can be thought of as vectors in the complex plane with tails at z_i and p_i and heads at ψ. Fig. A.01.3 shows this geometric interpretation with $p_{1,2} = -3 \pm j3$, $z_1 = 1$, and $\psi = 3 + j4$.

Example A.01–2 **re: transfer function mapping a contour**

Let $I = [0, 2\pi]$. Let the contour $\Gamma : I \to \mathbb{C}$ be defined parametrically, with $t \in I$, as

$$\Gamma(t) = \sin t + j \cos t.$$

Map Γ with the transfer function

$$H(s) = \frac{s}{s^2 + 2s + 2}$$

and plot the result.

```
\[CapitalGamma][t_] := {Sin[t], Cos[t]}
H[s_] := (s + 1)/(s^2 + 2*s + 2);

ps = {Blue, Arrowheads[{0, .05, .05, .05}]};
mappingcontour = Animate[
  {
    ParametricPlot[
      \[CapitalGamma][t], {t, 0, T},
      PlotRange -> {-1, 1},
      PlotStyle -> ps,
      PlotLabel -> "\[CapitalGamma]"
    ] /.
      Line -> Arrow,
    ParametricPlot[
      H[Complex @@ \[CapitalGamma][t]] // {Re[#], Im[#]} &,
      {t, 0.001, T},
      PlotRange -> {-1.5, 1.5},
      PlotStyle -> ps,
      PlotLabel -> "H(\[CapitalGamma])"
    ] /.
        Line -> Arrow
  } // GraphicsRow,
  {T, 0, 2*\[Pi]}
]
```

Figure A.01.5: a basic Mathematica script for visualizing the transfer function mapping of Example A.01-2. A more thorough notebook is available here.