rlocus.def Root locus definition

Closed-loop poles are hard to find

A feedback system's closed-loop poles determine its stability and transient response. These poles depend on parameters in the controller, often the gain K, as shown in Fig. def.1.

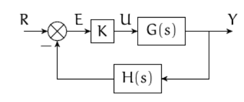

Figure def.1: a block diagram for a proportional controller K.

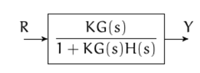

Figure def.2: a block diagram with the corresponding closed-loop transfer function block.

We define the forward transfer function to be the transfer function from a loop's error E to its output Y; for the loop in Fig. def.2, this is simply $KG(s)$. Furthermore, the feedback transfer function is defined to be the to be the transfer function from the output to the feedback summation, $H(s)$ in the example. Finally, the open-loop transfer function is defined to be the product of the two—in the case of our example: $KG(s)H(s)$.
We break down $G(s)$ and $H(s)$ into numerators and denominators:

$$G(s) = \frac{G_n(s)}{G_d(s)} \quad \text{and} \quad H(s) = \frac{H_n(s)}{H_d(s)}. \tag{1}$$

Now we can see how these affect the closed-loop transfer function

$$\frac{KG_n(s)H_d(s)}{G_d(s)H_d(s) + KG_n(s)H_n(s)}. \tag{2}$$

We make the following observations:

1. The closed-loop poles depend on K, but for controller design, K is that for which we are solving.
   a) An analytic solution for the closed-loop poles is intractable for systems of order greater than three.
   b) For a given value of K, a numerical root finder is very effective.

   That is, the closed-loop poles are hard to find!
2. As $K \to 0$, the closed-loop poles approach the open-loop poles.
3. As $K \to \infty$, the closed-loop poles approach the open-loop zeros.

These last two observations give us the "start" and "finish" for closed-loop pole locations when K is varied from 0 to ∞, a procedure we will now define—as the root locus!

Definition

The root locus is the collection of closed-loop pole locations for varying proportional controller gain K. Recall that for a feedback system with plant $G(s)$ and feedback transfer function $H(s)$, the closed-loop transfer function is

$$\frac{KG(s)}{1 + KG(s)H(s)} \tag{3}$$

and that finding the poles is difficult, in general. However, let us consider our observations from Lec. rlocus.def. We know our "starting points": at $K = 0$, the closed-loop poles are equal to the open-loop poles. And we know our "end points": as $K \to \infty$, the closed-loop poles are equal to the open-loop zeros. Therefore, we can consider the root locus to be a collection of curves that begin at the open-loop poles and terminate at the open-loop zeros.

The magnitude and phase criteria

The closed-loop poles can be found by setting the denominator of the closed-loop transfer function to zero and solving for the values of s that satisfy this condition. Examining the closed-loop transfer function, we see this is equivalent to

$$1 + K\,G(s)\,H(s) = 0$$
$$K\,G(s)\,H(s) = -1 = 1\angle(1+2m)\pi \tag{4}$$

Eq. 4 gives rise to the magnitude and phase criteria.

Equation 5 magnitude criterion

$$|K\,G(s)\,H(s)| = 1$$
for all s

Equation 6 phase criterion

$$\angle K\,G(s)\,H(s) = \angle(1+2m)\pi$$
for all s $\quad m \in \mathbb{Z}$

These criteria are always satisfied and so they will be our guide to understanding, sketching, and designing with the root locus.

What about negative gains and positive feedback?

We typically consider only nonnegative gains for the root locus, since negative gains typically lead to instability. However, this is only true for systems with positive open-loop transfer functions! When encountering a negative open-loop transfer function, it is advisable to temporarily treat it as positive, proceed with the controller design, then multiply the controller by -1 (or use positive feedback). If one suspects a negative gain might be of service in a specific controller (often, slightly negative gains can remain stable) or if one is building an unstable system intentionally, develop the root locus for negative gains (or, equivalently, positive feedback); for these occasions, see N. Nise (2015).