freq.freqtime Relations among time and frequency domain reps

The second-order assumption

As in root locus design, our transient response characteristics—such as percent overshoot $\%OS$, settling time T_s, and peak time T_p—can be related exactly to second-order response characteristics ζ and ω_n, which have their own interpretations in the frequency domain and are related to key features of the Bode plot. This often gets us close enough that small iterations on the initial design can achieve the desired transient response.
The second-order approximation assumes an open-loop transfer function of the form

$$G(s) = \frac{\omega_n^2}{s(s+2\zeta\omega_n)} \tag{1}$$

which yields a closed-loop transfer function

$$T(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}, \tag{2}$$

which has a familiar frequency response.

Bandwidth

The term bandwidth appears in many contexts, but in control theory when using the second-order assumption it has a very specific definition.

Definition freq.1: bandwidth

Let a system have a transfer function $G(s)$ and frequency response function $G(j\omega)$. The bandwidth ω_{BW} of the system is the angular frequency at which $|G(j\omega)|$ is $3\,\text{dB}$ less than $|G(j0)|$.

Closed-loop percent overshoot from the closed-loop bandwidth

It is straightforward to show that the bandwidth of the second-order closed-loop transfer of Eq. 2 is related to its natural frequency ω_n and damping ratio ζ by the expression

$$\omega_{BW} = \omega_n\left((1-2\zeta^2)+(4\zeta^4-4\zeta^2+2)^{1/2}\right)^{1/2}. \tag{3}$$

So if a system behaves approximately like this second-order system, Eq. 3 relates the closed-loop frequency response characteristic ω_{BW} and the closed-loop time response characteristics ω_n and ζ. This is a big step, but we often design the speed of response in terms of settling time T_s, peak time T_p, and rise time T_r. We already have relationships for these quantities and ω_n and ζ, the consequences of two of which when applied to Eq. 3 are shown below:

$$\omega_{BW} = \frac{4}{T_s\zeta}\left((1-2\zeta^2)+(4\zeta^4-4\zeta^2+2)^{1/2}\right)^{1/2} \tag{4a}$$

$$\omega_{BW} = \frac{\pi}{T_p\sqrt{1-\zeta^2}}\left((1-2\zeta^2)+(4\zeta^4-4\zeta^2+2)^{1/2}\right)^{1/2}. \tag{4b}$$

Furthermore, it can be shown that, when it exists (which it does for $0<\zeta<1/\sqrt{2}$), the peak magnitude $M_p = \max_\omega |H(j\omega)|$ is

$$M_p = \frac{1}{2\zeta\sqrt{1-\zeta^2}}. \tag{5}$$

Of course, percent overshoot $\%OS$ is directly related to ζ by the equations

$$\%OS = 100\exp\frac{-\zeta\pi}{\sqrt{1-\zeta^2}} \iff \zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2+\ln^2(\%OS/100)}} \tag{6}$$

so M_p can be directly related to $\%OS$.

Closed-loop percent overshoot and damping ratio from the open-loop phase margin

From closed-loop considerations of the frequency response, we have learned to determine some closed-loop time response characteristics. Now we learn to determine one of these characteristics—percent overshoot $\%OS$—from open-loop frequency response. From the transfer function of Eq. 1, it is straightforward to relate the phase margin Φ_M of the open-loop transfer function to the damping ratio ζ via the expressions

$$\Phi_M = \arctan\frac{2\zeta}{\sqrt{-2\zeta^2+\sqrt{1+4\zeta^4}}} \iff \tag{7}$$

$$\zeta = \frac{\tan\Phi_M}{2(1+\tan^2\Phi_M)^{1/4}}. \tag{8}$$

As we know from Eq. 6, percent overshoot $\%OS$ is directly related to ζ, so Φ_M can be directly related to $\%OS$, as shown in Fig. freqtime.1.

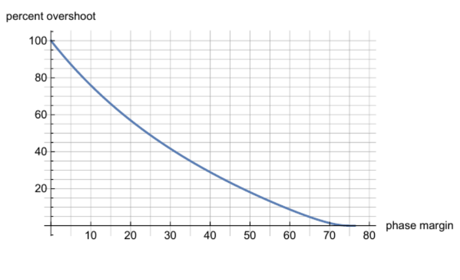

Figure freqtime.1: percent overshoot $\%OS$ versus phase margin Φ_M.

Closed-loop settling and peak times from the open-loop frequency response

We introduced the concept of bandwidth in above and related the closed-loop bandwidth ω_{BW} to settling time T_s and peak time T_p in Eq. 4. There is a method, which we present but leave underived, that allows us to find the closed-loop bandwidth of many systems from the open-loop frequency response, allowing us to relate the open-loop frequency response to T_s and T_p. The method is based on the following insight: the closed-loop bandwidth is approximately equal to the frequency at which the magnitude of the open-loop frequency response is in the interval $[-6,-7.5]$ dB if the phase of the open-loop frequency response is in the interval $[-135,-225]$ deg.
This gives us a method to approximate closed-loop T_s and T_p by inspecting the open-loop frequency response. Here's the method:

1. estimate the closed-loop bandwidth ω_{BW} by finding the frequency at which the magnitude of the open-loop frequency response is in the interval $[-6,-7.5]$ dB;
2. verify that open-loop phase at ω_{BW} is in the interval $[-135,-225]$ deg;
3. determine ζ via the phase margin (Eq. 7, Fig. freqtime.2); and
4. estimate T_s and T_p via Eq. 4.

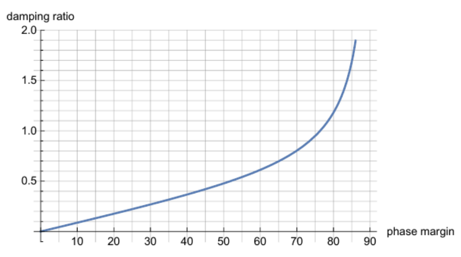

Figure freqtime.2: damping ratio ζ versus phase margin Φ_M.