B.02 Canonical forms of the state model

There are several canonical forms for the state equations, all of which can be found via basis transformations from other forms.

Phase-variable canonical form

The phase-variable canonical form is represented by the SISO[1] state model

$$\dot{x}_c = A_c x_c + B_c u \tag{1a}$$
$$y = C_c x_c + D_c u \tag{1b}$$

where

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix}, \quad B_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix}, \tag{1c}$$

$$C_c = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}, \text{ and} \qquad D_c = \begin{bmatrix} d_1 \end{bmatrix}. \tag{1d}$$

1. There are phase-variable canonical forms for MIMO systems as well, but these are less standardized.

In order to transform a SISO system $\{A, B, C, D\}$ with state vector x to phase-variable canonical form, we change bases via the substitution of $x = T_c x_c$ into the original system, which gives

$$A_c = T_c^{-1} A T_c, \qquad B_c = T_c^{-1} B, \tag{2a}$$
$$C_c = C T_c, \text{ and} \qquad D_c = D. \tag{2b}$$

The special form of Equation 1 yields the following characteristic polynomial:

$$s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0. \tag{3}$$

Recall that eigenvalues of a system are invariant to basis change, and therefore so is its characteristic polynomial. From this we can conclude that A_c can be completely determined by finding the characteristic polynomial of the original matrix A. B_c is already fully determined, but C_c and D_c remain undetermined. They may be found by discovering the transformation matrix T_c and substituting it into Equation 2.

Finding the phase-variable canonical transformation

The phase-variable canonical transformation matrix T_c can be found by relating the controllability matrices of the original form and the canonical form.

Theorem B.4: phase-variable canonical transformation

he transformation matrix from a system representation with controllability matrix $\mathcal{U}$ to a phase-variable canonical transformation with controllability matrix $\mathcal{U}_c$ is

$$T_c = \mathcal{U}_c \mathcal{U}^{-1}. \tag{4}$$

By the Definition of the controllability matrix, the original controllability matrix is

$$\mathcal{U} = \left[B \,|\, AB \,|\, A^2 B \,|\, \ldots \,|\, A^{n-1} B \right] \tag{5}$$

and that of the canonical form is

$$\mathcal{U}_c = \left[B_c \,|\, A_c B_c \,|\, A_c^2 B_c \,|\, \ldots \,|\, A_c^{n-1} B_c \right]. \tag{6}$$

Note that $\mathcal{U}$ and $\mathcal{U}_c$ are both known from above. We relate the two forms by applying Equation 2 to Equation 6 to yield

$$\mathcal{U}_c = \left[T_c^{-1} B \,|\, T_c^{-1} AB \,|\, T_c^{-1} A^2 B \,|\, \ldots \,|\, T_c^{-1} A^{n-1} B \right] \tag{7a}$$
$$= T_c \mathcal{U}, \tag{7b}$$

to yield

$$T_c = \mathcal{U}_c \mathcal{U}^{-1}.$$