ss.sfdbck Controller design method

We will consider single-input single-output (SISO) control plants that can be written with input u; state vector $\mathbf{x}$; output y; state model matrices A, B, C, and D; and state and output equations

$$\dot{\mathbf{x}} = A\mathbf{x} + Bu \tag{1a}$$
$$y = C\mathbf{x} + Du. \tag{1b}$$

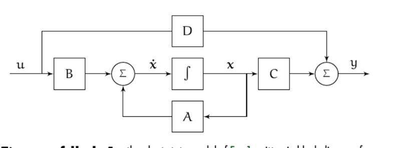

Figure sfdbck.1: the plant state model of Eq. 1 written in block diagram form.

Plants of this form can be written in block diagram form, as illustrated in Fig. sfdbck.1. In general, SISO systems are of order n with n state variables.

Let us consider the following feedback control method called state feedback control. We will feed back the state vector $\mathbf{x}$, operate on it with a $1 \times n$ vector of gains $\mathbf{K} \in \mathbb{R}^n$, and subtract the result from the command r, the result of which becomes the input u, as shown in Fig. sfdbck.2. The control problem for state feedback control is to determine the n gains in $\mathbf{K}$ such that the closed-loop poles are located in desirable positions. The gain $N \in \mathbb{R}$ is provided for steady-state error considerations, which will be addressed in Lec. ss.sfdbck. A new state model can be derived for the closed-loop system as follows. Let us consider the command r to be our new "input," instead of u, which is now the control effort. From the block diagram,

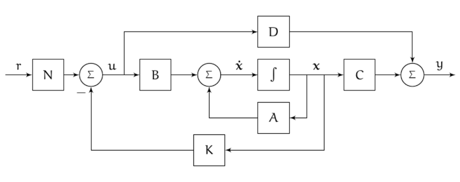

Figure sfdbck.2: the state feedback control block diagram.

$$u = Nr - K\mathbf{x}, \tag{2}$$

which can be substituted into Eq. 1 to define the new state model

$$\dot{\mathbf{x}} = (A - BK)\mathbf{x} + NBr \tag{3a}$$
$$y = (C - DK)\mathbf{x} + NDr. \tag{3b}$$

$$\dot{x} = Ax + Bu$$
$$= Ax + B(Nr - Kx)$$
$$= Ax - BKx + BNr$$
$$= (A - BK)x + BNr$$

The eigenvalues of $A - BK$, which can be found from equating zero and the closed-loop characteristic polynomial

$$P_K = \det(sI - A + BK), \tag{4}$$

$$H(s) = (C - DK)(sI - A + BK)^{-1} NB + ND$$

are equal to the closed-loop poles, which we would like to place in specific locations. Those specific locations can be specified by the design characteristic polynomial P_d. P_K depends on the n gains K_i, and n equations can be found by equating the polynomial coefficients of P_K and P_d.

Solving for K_i is straightforward but can be very tedious in the general case. Let the coefficients of P_d be δ_i and those of P_K be denoted κ_i. Then the $n \times 1$ vector containing κ_i can be expressed as a linear combination of K_i as

$$\boldsymbol{\kappa} = \mathcal{K}\,\mathbf{K}^\top, \tag{5}$$

where $\mathcal{K}$ is an $n \times n$ matrix of coefficients that were derived from A and B. Let $\boldsymbol{\delta}$ be the $n \times 1$ vector of components δ_i. Since the vector $\boldsymbol{\delta}$ is specified by our design requirements, we can solve for $\mathbf{K}$ as follows.

$$\boldsymbol{\kappa} = \boldsymbol{\delta}, \tag{6}$$

and therefore,

$$\begin{aligned} \mathcal{K}\,\mathbf{K}^\top &= \boldsymbol{\delta} \implies \\ \mathbf{K}^\top &= \mathcal{K}^{-1}\boldsymbol{\delta} \implies \\ \mathbf{K} &= (\mathcal{K}^{-1}\boldsymbol{\delta})^\top. \end{aligned} \tag{7}$$

Eq. 7 is valid for all cases in which $\mathcal{K}$ is invertible.[1] However, there is a special form of the original state-space model that always yields a simple solution for $\mathbf{K}$: the phase-variable canonical form (see Appendix B.02).

1. We leave the following as an open question: under what conditions is $\mathcal{K}$ invertible?

Solving for the gain via the phase-variable canonical form

The phase-variable canonical form of the original system is:

$$\dot{\mathbf{x}}_c = A_c\mathbf{x}_c + B_c u \tag{8a}$$
$$y = C_c\mathbf{x}_c + D_c u \tag{8b}$$

where

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix}, \quad B_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix}, \tag{8c}$$

$$C_c = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}, \text{ and} \quad D_c = \begin{bmatrix} d_1 \end{bmatrix}, \tag{8d}$$

where the components a_i are defined by the original characteristic polynomial

$$P = \det(sI - A) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0. \tag{9}$$

With A_c defined, the form of the feedback state model with feedback row vector $\mathbf{K}_c$ is:

$$A'_c = A_c - B_c\mathbf{K}_c, \qquad B'_c = B_c, \tag{10a}$$
$$C'_c = C_c - D_c\mathbf{K}_c, \text{ and} \qquad D'_c = D_c. \tag{10b}$$

A'_c deserves further attention. The special canonical form of A_c and B_c makes the expression for A'_c simply

$$A'_c = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & 0 \\ 0 & 0 & \cdots & 1 \\ -(a_0 + K'_1) & -(a_1 + K'_2) & \cdots & -(a_{n-1} + K'_n) \end{bmatrix}, \tag{11}$$

where K'_i is the row vector of gains in the phase-variable canonical basis. The design characteristic polynomial coefficients δ_i must equal the characteristic polynomial coefficients

$$\delta_i = a_i + K'_{i+1}, \tag{12}$$

which gives

$$K'_i = \delta_{i-1} - a_{i-1}. \tag{13}$$

This yields $\mathbf{K}'$. If we equate the feedback

$$\begin{aligned} \mathbf{K}\mathbf{x} &= \mathbf{K}'\mathbf{x}_c \implies \\ \mathbf{K} &= \mathbf{K}'T_c. \end{aligned} \tag{14}$$

Let $\mathcal{U}$ and $\mathcal{U}_c$ be the controllability matrices for the original basis and the phase-variable canonical basis, respectively. From Appendix B.02, we can compute the transformation matrix to be

$$T_c = \mathcal{U}_c\,\mathcal{U}^{-1}. \tag{15}$$

Steady-state error

We can use the gain N to drive the closed-loop steady-state error to zero for step inputs. The idea is that we can scale the input by the reciprocal of the closed-loop steady-state error. Let $G_{CL}(s)$ be the closed-loop transfer function. From the final value theorem for a unit step input,

$$N = \lim_{s \to 0,\, N \to 1} 1/G_{CL}(s). \tag{16}$$

If N is nonzero and finite, the response will have zero steady-state error. Although it is derived from unit step inputs, we can apply this formula to slowly varying inputs as well.

Example ss.sfdbck−1 **re: state feedback pole placement design**

Given the state-space model

$$A = \begin{bmatrix} -1 & 0 & -1 \\ -1 & -1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$
$$C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 0 \end{bmatrix},$$

design a controller with 15% overshoot and a settling time of 1 sec.

$$\zeta = \frac{-\ln(15/100)}{\sqrt{\pi^2 + \ln^2(15/100)}} = \frac{1.897}{3.67} = 0.52$$

$$T_s = \frac{4}{\zeta\omega_n} \Rightarrow \omega_n = \frac{4}{\zeta T_s} = \frac{4}{0.52} = 7.74 \text{ rad/s}$$

$$P_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2} = -4 \pm 6.6j$$

$$H(s) = C(sI - A)^{-1}B + D = \frac{1}{s^3 + 3s^2 + 3s + 2}$$

$$P_3 = -20$$

$$P_d = (s - p_1)(s - p_2)(s - p_3) = s^3 + 28s^2 + 219.9s + 1198$$

$$A_c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -3 & -3 \end{bmatrix} \qquad B_c = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$K' = [1198 - 2,\ 219.9 - 3,\ 28 - 3]$$
$$= [1196,\ 216.9,\ 25]$$

$$B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad AB = \begin{bmatrix} -1 & 0 & -1 \\ -1 & -1 & 0 \\ 0 & -1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} \qquad A^2B = AAB = \begin{bmatrix} -1 & 0 & -1 \\ -1 & -1 & 0 \\ 0 & -1 & -1 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\mathcal{U} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B_c = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad A_cB_c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -3 & -3 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} \qquad A_c^2B_c = A_cA_cB_c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -3 & -3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 6 \end{bmatrix}$$

$$\mathcal{U}_c = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & -3 & 6 \end{bmatrix} \qquad T_c = \mathcal{U}_c\,\mathcal{U}^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & -3 & 6 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & 2 & 1 \end{bmatrix}$$

$$K = K'T_c = \begin{bmatrix} 1196 & 216.9 & 25 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & 2 & 1 \end{bmatrix} = [25,\ -166.9,\ 1004.1]$$

$$N = \lim_{\substack{s \to 0 \\ N \to 1}} \frac{1}{N(C - DK)(sI - A + BK)^{-1}B + ND}$$
$$= \frac{1}{(C - DK)(-A + BK)^{-1}B + D} = 1198$$