Optimal Control

most control design methods are for SISO
optimal control MIMO systems

$\dot{x} = Ax + Bu$ $\qquad$ $y = Cx + Du$

$$J = \int_0^\infty x^T Q x + u^T R u \, dt$$

Q, R symmetric
$Q \geq 0$ PSD
$R > 0$ PD

find u that minimizes J

Hamilton Jacobi Bellman equation

$$0 = \min_u x^T Q x + u^T R u + \frac{\partial J^*}{\partial x}(Ax + Bu)$$

$$J^*(x) = x^T S x \qquad S \text{ symmetric}$$

$$\frac{\partial J^*}{\partial x} = 2x^T S \qquad S \geq 0$$

$$0 = \min_u x^T Q x + u^T R u + 2x^T S(Ax + Bu)$$

$$\frac{\partial}{\partial u} = 2u^T R + 2x^T SB = 0$$

$$2u^T R = -2x^T SB$$

$$u^T R = -x^T SB$$

$$u^T = -x^T SBR^{-1}$$

$$u = -R^{-1}B^T S x = -Kx$$

$$K = R^{-1}B^T S$$

$$0 = x^T Q x + x^T SBR^{-1}RR^{-1}B^T S x + 2x^T S(Ax - BR^{-1}B^T S x)$$

$$0 = x^T Q x + x^T SBR^{-1}B^T S x + 2x^T SAx - 2x^T SBR^{-1}B^T S x$$

$$0 = x^T Q x - x^T SBR^{-1}B^T S x + 2x^T SAx$$

$$0 = x^T(Q - SBR^{-1}B^T S + 2SA)x$$

$$0 = Q - SBR^{-1}B^T S + 2SA$$

$$0 = Q - SBR^{-1}B^T S + SA + A^T S \qquad \text{Ricatti Eqn}$$

Multiple Input Multiple Output Systems

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix} \qquad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \end{bmatrix} \qquad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \end{bmatrix}$$

$\dot{x} = Ax + Bu$

SISO

$$\dot{x} = Ax + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \end{bmatrix} \begin{bmatrix} u \end{bmatrix}$$

MIMO

$$\dot{x} = Ax + \begin{bmatrix} b_{11} & b_{21} & \cdots \\ b_{12} & b_{22} & \cdots \\ \vdots & \vdots & \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \end{bmatrix}$$

$y = Cx + Du$

SISO

$$y = \begin{bmatrix} c_1 & c_2 & \cdots \end{bmatrix} x$$

MIMO

$$y = \begin{bmatrix} c_{11} & c_{12} & \cdots \\ c_{21} & c_{22} & \cdots \\ \vdots & & \end{bmatrix} x$$