Kalman Filters

Markov Localization

Pros

- Works well
- Simple
- Arbitrary/Multimodal PDFs

Cons

- Must store PDF of entire map

Kalman Filter

Pros

- Fewer Calculation
- Can deal with arbitrarily large environment

Cons

- Assume Gaussian Distribution
- Unimodal PDF

Example

SMU Campus

380 Acres $\quad$ 1 Acre $\simeq$ 4047 m^2

1.54×10^6 m^2

1240 m^2

0.5 m resolution

2480 X 2480 array

6.15×10^6 elements

Kalmon Filter

Optimal Estimator

Uses gaussian distributions

Discrete state based systems

What is an estimator?

Uses a series of measurements to estimate the state vector

Observability Matrix

$\dot{x} = Ax + Bu$ $\qquad$ $y = Cx + Du$

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

if $\operatorname{rank}(\mathcal{O}) = n$

if $\mathcal{O}$ is invertible

Example

$$\begin{bmatrix} \dot{v} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} -\frac{Br}{m} & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v \\ x \end{bmatrix} + \begin{bmatrix} \frac{k_m r}{m} \\ 0 \end{bmatrix} \begin{bmatrix} I_s \end{bmatrix}$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ x \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} \begin{bmatrix} I_s \end{bmatrix}$$

$$= \begin{bmatrix} x \end{bmatrix}$$

$$y = Cx + Du$$

$$O = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$CA = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{Br}{m} & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$|O| = -1$$

Example

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} u_1 \end{bmatrix}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} \begin{bmatrix} u_1 \end{bmatrix}$$

$$O = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

$$CA = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$|O| = 0$$

$$x_t = A_d x_{t-1} + B_d u_t + w_t \qquad w_t \sim N(0, Q)$$

w_t prcess noise

$$z_t = H x_t + v_t \qquad v_t \sim N(0, R)$$

v_t measurement noise

$\hat{x}$ estimate of true state x

$\hat{x}_{t|t}$ estimate at time t using all measurements up to time t

$\hat{x}_{t|t-1}$ estimate at time t using all measurements up to time $t-1$

Predict

$$\hat{X}_{t|t-1} = A_d \hat{x}_{t-1|t-1} + B_d a_t$$

$$P_{t|t-1} = A_d P_{t-1|t-1} A_d^T + Q$$ P variance of the estimate

Update

$$\hat{x}_{t|t} = \hat{x}_{t|t-1} + K y_t$$

$$P_{t|t} = (I - KH) P_{t|t-1}$$

$$y_t = z_t - H\hat{x}_{t|t-1}$$ measurement residual

K Kalman Gain

$\tilde{x}_{t|t} = x_t - \hat{x}_{t|t}$ state residual

$\min \ E[\|\tilde{x}_{t|t}\|_2]$

$tr\left(\begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \\ \vdots & & \ddots \end{bmatrix}\right) = a_{11} + a_{22} + \cdots + a_{nn}$

$\min_{K} \ tr(P_{t|t})$

$P_{t|t} = (I - KH)P_{t|t-1}(I-KH)^T + KRK^T$

$= P_{t|t-1} - KHP_{t|t-1} - P_{t|t-1}H^TK^T + KHP_{t|t-1}H^TK^T + KRK^T$

$= P_{t|t-1} - KHP_{t|t-1} - P_{t|t-1}H^TK^T + KSK^T$

$S = HP_{t|t-1}H^T + R$

$\frac{\partial tr(P_{t|t})}{\partial K} = -2(HP_{t|t-1})^T + 2KS = 0$

$(HP_{t|t-1})^T = KS$

$P_{t|t-1}^T H^T = KS$

$P_{t|t-1}^T H^T S^{-1} = K$

$$\left.\begin{aligned}
\hat{x}_{t|t-1} &= A_d \hat{x}_{t-1|t-1} + B_d u_t \\
P_{t|t-1} &= A_d P_{t-1|t-1} A_d^T + Q
\end{aligned}\right\} \text{Predict}$$

$$\left.\begin{aligned}
y_t &= Z_t - H\hat{x}_{t|t-1} \\
S &= H P_{t|t-1} H^T + R \\
K &= P_{t|t-1}^T H^T S^{-1} \\
\hat{x}_{t|t} &= \hat{x}_{t|t-1} + K y_t \\
P_{t|t} &= (I - KH) P_{t|t-1}
\end{aligned}\right\} \text{Update}$$

Extended Kalman Filter

$$x_t = f(x_{t-1}, u_t) + w_t \qquad w_t \sim N(0, Q)$$

$$z_t = h(x_t) + v_t \qquad v_t \sim N(0, R)$$

Predict

$$\hat{x}_{t|t-1} = f(\hat{x}_{t-1|t-1}, u_t)$$

$$P_{t|t-1} = F_t P_{t-1|t-1} F_t^T + Q$$

$$F_t = \left.\frac{\partial f}{\partial x}\right|_{\hat{x}_{t-1|t-1}, u_t}$$

if $f(x, u) = Ax + Bu$

$$\frac{\partial f}{\partial x} = A$$

Update

$$y = z_t - h(\hat{x}_{t|t-1})$$

$$S = H_t P_{t|t-1} H_t^T + R$$

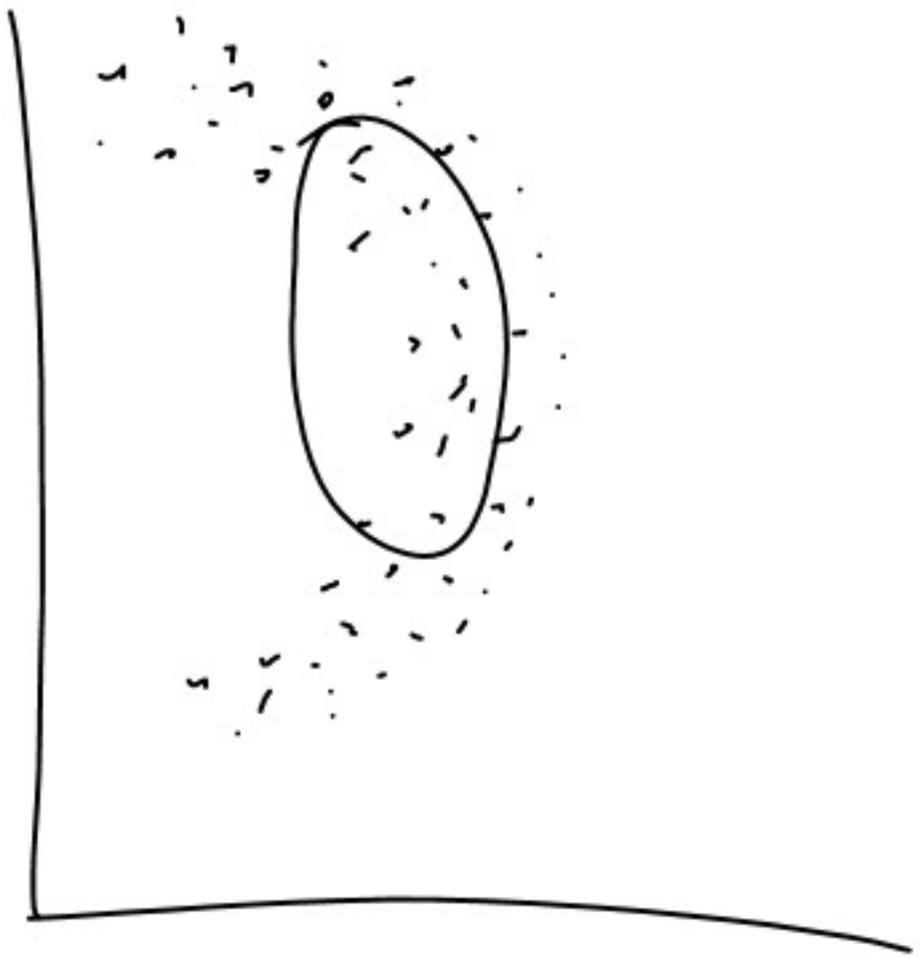
$$H_t = \left.\frac{\partial h}{\partial x}\right|_{\hat{x}_{t|t-1}}$$

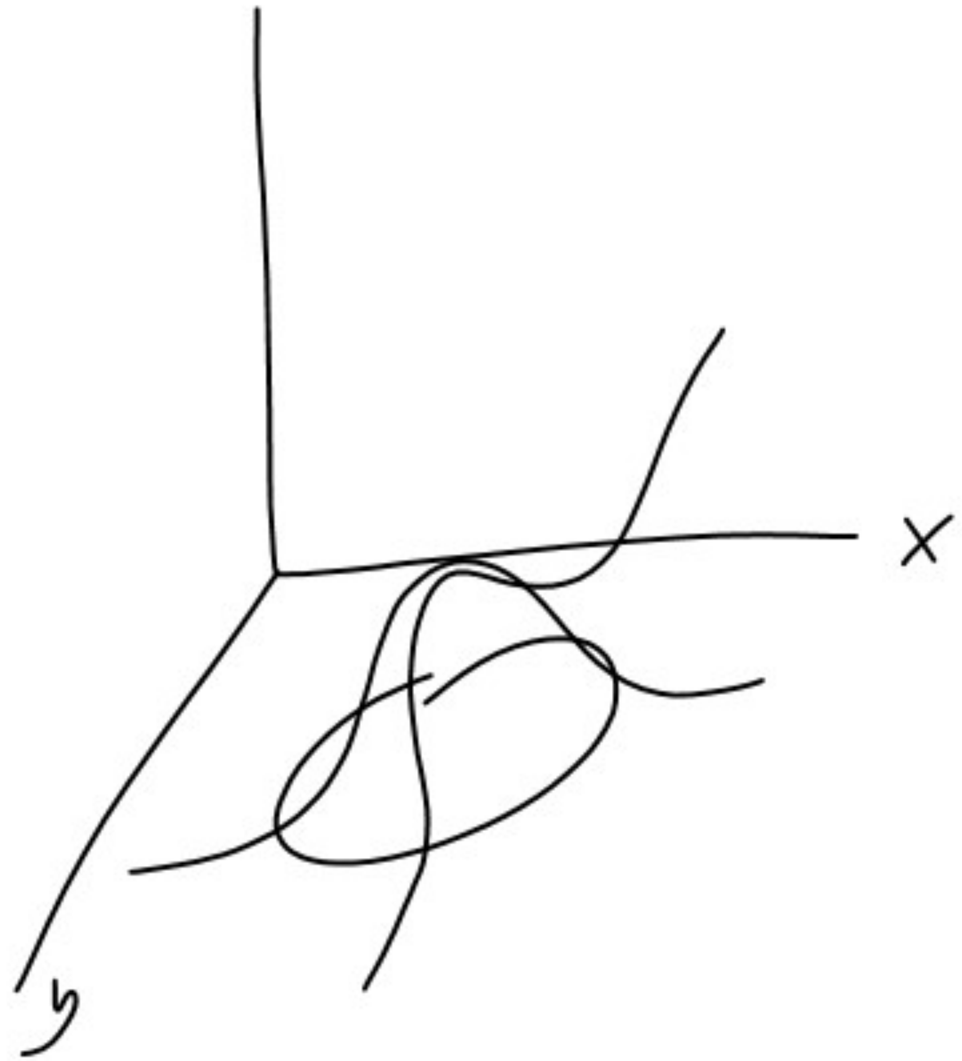
$$K = P_{t|t-1}^T H_t^T S^{-1}$$

$$\hat{x}_{t|t} = \hat{x}_{t|t-1} + Ky$$

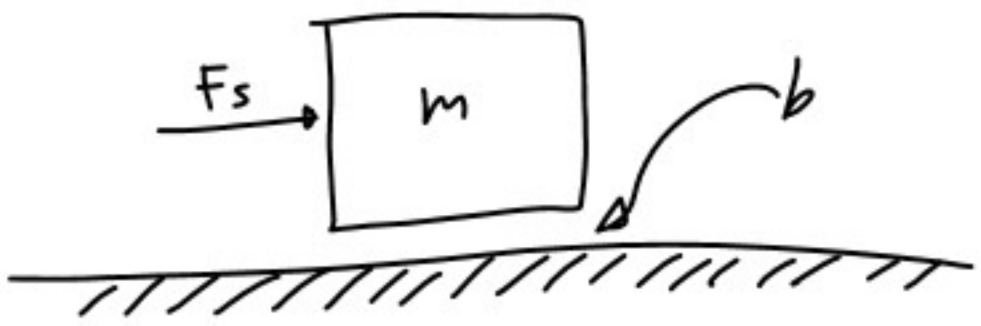
$$P_{t|t} = (I - KH_t) P_{t|t-1}$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \ddots \\ & & \end{bmatrix}$$




X
y

Example



$$x = [V_m] \qquad u = [F_s]$$

$$\dot{x} = [-b/m]\,x + [1/m]\,u$$

$$X = \begin{bmatrix} V_m \\ x_m \end{bmatrix}$$

$$\dot{x} = \begin{bmatrix} -b/m & 0 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1/m \\ 0 \end{bmatrix} u$$

$$A_d = I + dt\,A$$

$$B_d = dt\,B$$

$$Z = [x_m]$$

$$= Hx$$

$$= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} V_m \\ x_m \end{bmatrix}$$