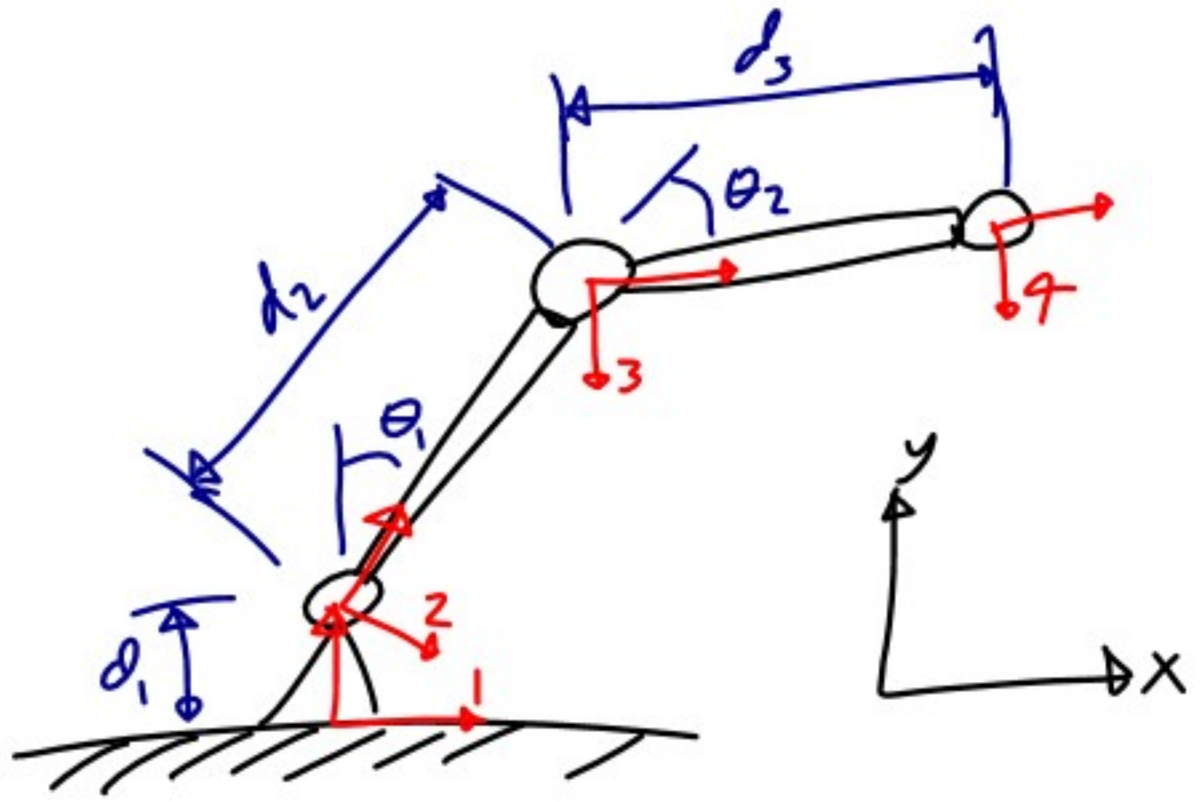


$${}^1_4T = {}^1_2T\,{}^2_3T\,{}^3_4T$$

$${}^1_4T = \begin{bmatrix} \begin{bmatrix} & & \\ & \mathrm{Rot}(z, \theta_1) & \\ & & \end{bmatrix} & \begin{matrix} 0 \\ d_1 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & 1 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} & & \\ & \mathrm{Rot}(z, \theta_2) & \\ & & \end{bmatrix} & \begin{matrix} 0 \\ d_2 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & d_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

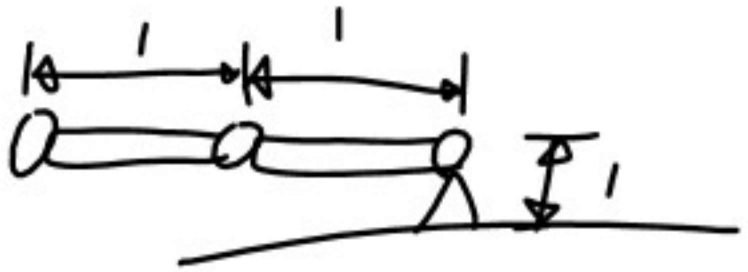
$$
{}^{1}_{4}T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & d_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_2 & -s_2 & 0 & 0 \\ s_2 & c_2 & 0 & d_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & d_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & d_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_2 & -s_2 & 0 & -d_3 s_2 \\ s_2 & c_2 & 0 & d_3 c_2 + d_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} c_1 c_2 - s_1 s_2 & -c_1 s_2 - s_1 c_2 & 0 & -d_3 c_1 s_2 - d_3 s_1 c_2 - d_2 s_1 \\ s_1 c_2 + c_1 s_2 & -s_1 s_2 + c_1 c_2 & 0 & -d_3 s_1 s_2 + d_3 c_1 c_2 + d_2 c_1 + d_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Forward Kinematics

$$f\left(\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix}\right)=\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}-d_2 s_1 - d_3 s_1 c_2 - d_3 c_1 s_2\\ d_1 + d_2 c_1 - d_3 s_1 s_2 + d_3 c_1 c_2\\ 0\end{bmatrix}$$



if $d_1 = d_2 = d_3 = 1$ $\quad \theta_1 = \pi/2 \quad \theta_2 = 0$

$$f\left(\begin{bmatrix}\pi/2\\ 0\end{bmatrix}\right)=\begin{bmatrix}-1\cdot 1 - 1\cdot 1\cdot 1 - 1\cdot 0\cdot 0\\ 1 + 1\cdot 0 - 1\cdot 1\cdot 0 + 1\cdot 0\cdot 1\\ 0\end{bmatrix}=\begin{bmatrix}-2\\ 1\\ 0\end{bmatrix}$$

Gimbal Lock

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_y & -S_y \\ 0 & S_y & C_y \end{bmatrix} \begin{bmatrix} C_p & 0 & S_p \\ 0 & 1 & 0 \\ -S_p & 0 & C_p \end{bmatrix} \begin{bmatrix} C_r & -S_r & 0 \\ S_r & C_r & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

pitch $\pi/2$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_y & -S_y \\ 0 & S_y & C_y \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} C_r & -S_r & 0 \\ S_r & C_r & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ S_y C_r + C_y S_r & -S_y S_r + C_y C_r & 0 \\ -C_y C_r + S_y C_r & C_y S_r + S_y C_r & 0 \end{bmatrix}$$

$$R = \begin{bmatrix} 0 & 0 & 1 \\ S(y+r) & C(y+r) & 0 \\ -C(y+r) & S(y+r) & 0 \end{bmatrix}$$

quaternions

$$a + bi + cj + dk$$

$$i^2 = j^2 = k^2 = -1$$

$$ij = -ji = k \qquad jk = -kj = i \qquad ki = -ik = j$$

$$ijk = -1$$

Inverse Kinematics

$$f^{-1}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$$

Jacaboian

$$J = \frac{\partial f}{\partial \theta} = \begin{bmatrix} \frac{\partial x}{\partial \theta_1} & \frac{\partial x}{\partial \theta_2} & \cdots \\ \frac{\partial y}{\partial \theta_1} & & \\ \vdots & & \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = J(\theta) \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$\underline{\dot{X}} = J\,\dot{\theta}$$

$$J^{-1}\underline{\dot{X}} = J^{-1}J\,\dot{\theta}$$

$$J^{-1}\underline{\dot{X}} = \dot{\theta}$$

$$J^{-1}\frac{\Delta \underline{X}}{\Delta t} = \frac{\Delta\theta}{\Delta t}$$

$$J^{-1}\Delta \underline{X} = \Delta\theta$$

$$J\left(\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix}\right) = \begin{bmatrix} -d_2 c_1 + d_3 s_1 s_2 - d_3 c_1 c_2 & d_3 s_1 s_2 - d_3 c_1 c_2 \\ -d_2 s_1 - d_3 s_1 c_2 - d_3 c_1 s_2 & -d_3 s_1 c_2 - d_3 c_1 s_2 \\ 0 & 0 \end{bmatrix}$$

Not square

$J^+ J = I$

Moore Penrose Pseudo Inverse

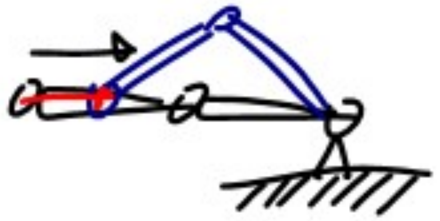
$A^{\dagger} = (A^{*}A)^{-1}A^{*}$ A^{*} complex conjugate transpose

$A^{\dagger}A = I$

$AA^{\dagger} \neq I$

A Potential Problem

$$\dot{X} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$



$$J\left(\begin{bmatrix} \pi/2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -d_2 \cdot 0 + d_3 \cdot 1 \cdot 0 - d_3 \cdot 0 \cdot 1 & d_3\, 1 \cdot 0 - d_3\, 0 \cdot 1 \\ -d_2 \cdot 1 - d_3\, 1 \cdot 1 - d_3\, 0 \cdot 1 & -d_3\, 1 \cdot 1 - d_3\, 0 \cdot 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ -d_2 - d_3 & -d_3 \\ 0 & 0 \end{bmatrix}$$

$$\dot{X} = J \dot{\theta}$$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -d_2 - d_3 & -d_3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$$

Singular Value Decomposition (SVD)

$M = U \Sigma V^*$

$M \in \mathbb{Z}^{m \times n}$

$U \in \mathbb{Z}^{n \times n}$

$\Sigma \in \mathbb{R}^{m \times n}$ diag all ≥ 0

$V \in \mathbb{Z}^{m \times m}$

$M^{\dagger} = V \Sigma^{\dagger} U^*$

$\Sigma^{\dagger}$ reciprical of all nonzero entries

Damped Least Squares

$$M^{+} = V(\Sigma^{+} + \lambda I)U^{*}$$