06.3 Discrete transfer functions

We begin with a review of Laplace transforms and continuous transfer functions.

Laplace transforms

In the analysis of this continuous systems, we use the Laplace transform, defined by

$$\mathcal{L}(f(t)) = \int_0^\infty f(t)e^{-st}dt \qquad (1)$$

which leads directly to the familiar Laplace transform properties (1) of linearity and (2) of differentiation: the Laplace transform of the derivative of a function $f(t)$ (with zero initial conditions) is s times the transform of the function $F(s) \equiv \mathcal{L}(f(t))$:

$$\mathcal{L}\left(\frac{df(t)}{dt}\right) = sF(s). \qquad (2)$$

Continuous transfer functions

These properties allow us to find the transfer function of a linear continuous system, given its differential equation. We define the continuous transfer function $T(s)$ to be the Laplace transform of the output $Y(s)$ divided by the Laplace transform of the input $X(s)$; i.e.

$$T(s) = \frac{Y(s)}{X(s)}. \qquad (3)$$

Reconsider the continuous differential equation for a dynamic system Eq. 1. The equivalent transfer function, using the linearity and differentiation properties of the Laplace transform, is

$$T(s) = \frac{\beta_m s^m + \beta_{m-1}s^{m-1} + \cdots + \beta_1 s^1 + \beta_0}{\alpha_n s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s^1_+ \alpha_0} \qquad (4)$$

where α_k and β_k are the same constants that appeared in Eq. 1.

z-Transforms

For discrete systems and their difference equations, a very similar procedure is available. The z-transform $F(z) \equiv \mathcal{Z}(f(n))$ of a sequence $f(n)$, with complex variable z (analogous to s), is defined by[3]

3. There are many more uses for z-transforms. For more details, see Franklin, Powell and Workman (1998).

$$\mathcal{Z}(f(n)) = \sum_{n=0}^{\infty} f(n)z^{-n}. \qquad (5)$$

This leads directly to the z-transform properties (1) of linearity and (2) of delay, analogous to (2) for discrete systems: the z-transform of a function delayed by one sample period is z^{-1} times the transform of the function $F(z)$:

$$\mathcal{Z}(f(n-1)) = z^{-1}F(z), \qquad (6)$$

Discrete transfer functions

We define the discrete transfer function $T(z)$ to be the z-transform of the output $Y(z)$ divided by the z-transform of the input $X(z)$; i.e.

$$T(z) = \frac{Y(z)}{X(z)}. \qquad (7)$$

Given the z-transform properties, we can easily find the transfer function of a discrete system given its difference equation.

Example 06.3 −1 **re: discrete transfer function**

What is the discrete tranfer function corresponding to the second-order difference equation

$$a_0 y(n) + a_1 y(n-1) + a_2 y(n-2) =$$
$$= b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) \qquad (8)$$

with constants a_n and b_n?

The z-transform of the difference equation is determined by linearity and successively applying (6) to arrive at

$Z(a_0 y(n) + a_1 y(n-1) + a_2 y(n-2))$

$Z(a_0 y(n)) + Z(a_1 y(n-1)) + Z(a_2 y(n-2))$

$a_0 Z(y(n)) + a_1 Z(y(n-1)) + a_2 Z(y(n-2))$

$a_0 Y(z) + a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z)$

$$(\cancel{1}_{a_0} + a_1 z^{-1} + a_2 z^{-2})\, Y(z) = (b_0 + b_1 z^{-1} + b_2 z^{-2})\, X(z). \qquad (9)$$

Rearranging, the discrete transfer function is

$$\frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{\cancel{1}_{a_0} + a_1 z^{-1} + a_2 z^{-2}} \qquad (10)$$

Notice that the transfer function (10) and the difference equation (8), can be derived from each other by inspection. Notice also that the transfer function of a discrete system is the ratio of two polynomials in z, just as the transfer function of a continuous system is the ratio of two polynomials in s.

Discrete approximations of continuous transfer functions

There are several ways to derive an approximate discrete transfer function from a corresponding continuous transfer function. We will use a popular technique called Tustin's method that approximates a continuous function of time by straight lines connecting the sampled points (i.e. trapezoidal integration). The discrete transfer function is found using Tustin's method by making the following substitution:

$$s \mapsto \frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right) \qquad (11)$$

and rewriting the transfer function in the form of equation (10). Here, T is the sample period.

Example 06.3 −2 **re: Tustin's method**

Consider a continuous first order system described by the transfer function:

$$\frac{Y(s)}{X(s)} = \frac{1}{\tau s + 1}, \text{ where } \tau \text{ is the time constant.} \qquad (12)$$

$$\frac{1}{\tau s + 1} \Longrightarrow \frac{1}{\tau \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 1} = \frac{1+z^{-1}}{\frac{2\tau}{T}(1-z^{-1}) + 1+z^{-1}}$$

Using Tustin's method, derive a discrete transfer function and the corresponding difference equation.

Substituting Equation 11 into the transfer function, we have:

$$\frac{Y(z)}{X(z)} = \frac{\alpha + \alpha z^{-1}}{1 - (1 - 2\alpha)z^{-1}},$$

where α is a constant:

$$\alpha = \frac{T}{2\tau + T}$$

from which the difference equation can be inferred (see Eqs. 8 to 10 above):

$$y(n) = \underbrace{(1 - 2\alpha)y(n-1)}_{\text{from denominator}} + \underbrace{\alpha x(n) + \alpha x(n-1)}_{\text{from numerator}},$$

Notice again that the current value of the output $y(n)$ depends on the previous output, $y(n-1)$, and on the current and previous inputs, $x(n)$ and $x(n-1)$.

Notice also that the coefficients depend on the time constant τ in the original continuous system and on the sample period T.

During each sample period, the value of the current value of the input $x(n)$ is measured and the current value of the output $y(n)$ is computed. Suppose that the time constant $\tau = 2$, the sample period $T = 1$, and that the input is a unit step ($x(n) = 1$ for all n), and the initial condition $y(0) = 0$.

$\alpha = \frac{T}{2\tau + T} = \frac{1}{2(2)+1} = \frac{1}{5} = 0.2$

$1 - 2\alpha = 1 - 2(.2) = 1 - 0.4 = 0.6$

Then, from our solution for $y(n)$,

$$y(n) = 0.6y(n-1) + 0.4 \qquad (13)$$

and we can compute the output sequence:

$y(0) = 0$
$y(1) = 0.4$
$y(2) = 0.64$
$y(3) = 0.78$
$y(4) = 0.87$

Figure 06.1 shows plots of the input and output sequences.

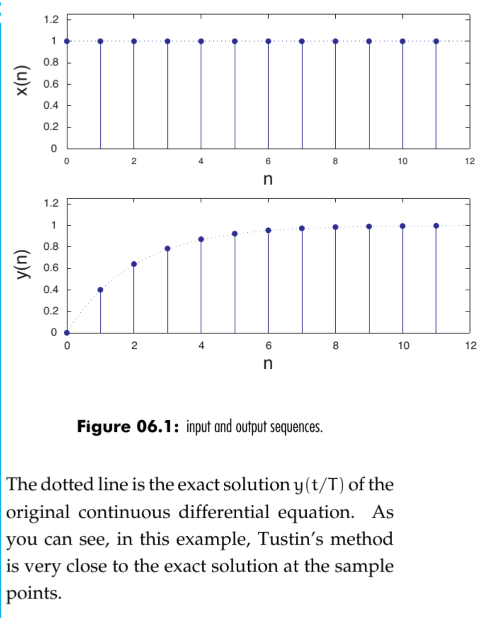

Figure 06.1: input and output sequences.

The dotted line is the exact solution $y(t/T)$ of the original continuous differential equation. As you can see, in this example, Tustin's method is very close to the exact solution at the sample points.

See Resource 13 for a table of common controller transfer functions converted to discrete transfer functions via Tustin's method.

Matlab's c2d

The Matlab Control Systems Toolbox includes a function c2d that computes the Tustin equivalent discrete system sysd from the continuous system sys, as follows.

ss
tf
zpk
continuous time

```
sysd = c2d(sys, T, 'tustin')
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This function can also use other common techniques to yield a discrete approximation of a continuous transfer function.