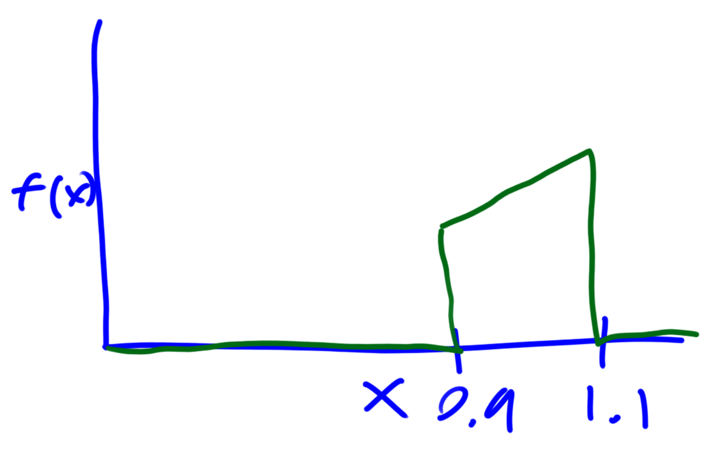
24.5

9. Let X [millimeters] be the thickness of washers. Assume that X has the density $f(x) = kx$ if $0.9 < x < 1.1$ and 0 otherwise. Find k. What is the probability that a washer will have thickness between 0.95 mm and 1.05 mm?



$$P(x \in [-\infty, \infty]) = \int_{-\infty}^{\infty} kx\,dx$$

$$P(x \in [0.9, 1.1]) =$$

$$\int_{0.9}^{1.1} kx\,dx = \frac{kx^2}{2}\Big|_{0.9}^{1.1} = 1$$

$$\frac{k(1.1)^2}{2} - \frac{k(0.9)^2}{2} = 1$$

$$k\left(\frac{(1.1)^2 - (0.9)^2}{2}\right) = 1$$

$$k = \frac{2}{(1.1)^2 - (0.9)^2} = \boxed{5}$$

$$P(x \in [0.95, 1.05])$$

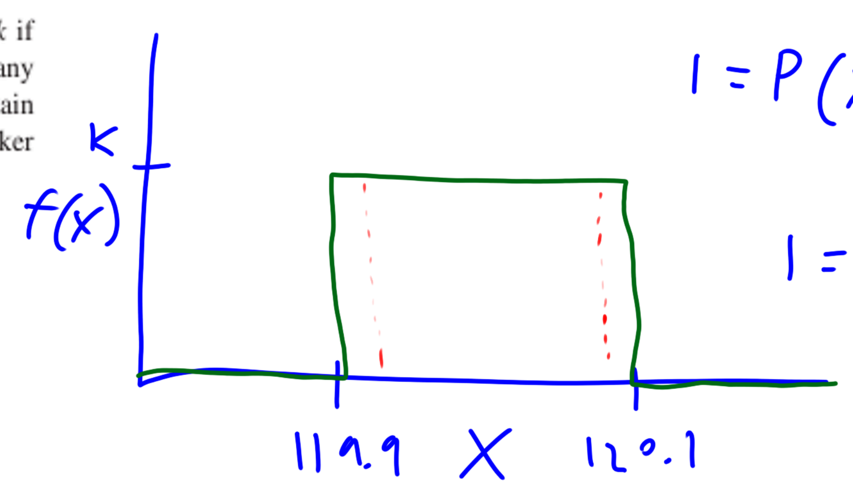
$$= \int_{0.95}^{1.05} 5x\,dx$$

$$= 5\,\frac{x^2}{2}\Big|_{0.95}^{1.05}$$

$$= \frac{5}{2}\left((1.05)^2 - (0.95)^2\right)$$

$$= \boxed{0.5}$$

10. If the diameter X of axles has the density $f(x) = k$ if $119.9 \leqq x \leqq 120.1$ and 0 otherwise, how many defectives will a lot of 500 axles approximately contain if defectives are axles slimmer than 119.91 or thicker than 120.09?



$$1 = P(X \in [-\infty, \infty]) = \int_{-\infty}^{\infty} f(x)\,dx$$

$$1 = A = k(120.1 - 119.9)$$

$$1 = k\,0.2$$

$$k = \frac{1}{0.2} = \boxed{5}$$

$$1 - P(x \in [119.91, 120.09])$$

$$1 - 5(120.09 - 119.91) = 1 - 5(0.18) = 1 - 0.9$$

$$= 0.1$$

$$500(0.1) = \boxed{50}$$

24.6

1–8 MEAN, VARIANCE

Find the mean and variance of the random variable X with probability function or density $f(x)$.

1. $f(x) = kx$ $(0 \leqq x \leqq 2$, k suitable)
2. X = Number a fair die turns up
3. Uniform distribution on $[0, 2\pi]$
4. $Y = \sqrt{3}(X - \mu)/\pi$ with X as in Prob. 3
5. $f(x) = 4e^{-4x}$ $(x \geqq 0)$
6. $f(x) = k(1 - x^2)$ if $-1 \leqq x \leqq 1$ and 0 otherwise
7. $f(x) = Ce^{-x/2}$ $(x = 0)$
8. X = *Number of times a fair coin is flipped until the first Head appears.* (Calculate μ only.)
9. If the diameter X [cm] of certain bolts has the density $f(x) = k(x - 0.9)(1.1 - x)$ for $0.9 < x < 1.1$ and 0 for other x, what are k, μ, and σ^2? Sketch $f(x)$.

1)

$$1 = \int_0^2 kx\,dx = \frac{kx^2}{2}\Big|_0^2 = \frac{k\,2^2}{2} = 2k$$

$$1 = 2k \implies k = \frac{1}{2}$$

$$E[x] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^2 x\,kx\,dx$$

$$= \frac{1}{2}\int_0^2 x^2\,dx = \frac{1}{2}\,\frac{x^3}{3}\Big|_0^2$$

$$= \frac{1}{2}\,\frac{2^3}{3} = \frac{2^2}{3} = \boxed{\frac{4}{3}}$$

$$Var[x] = \int_{-\infty}^{\infty} (x - E[x])^2 f(x)\,dx$$

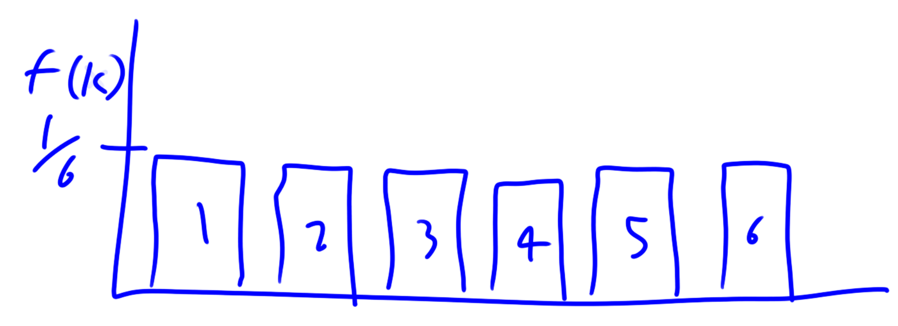
$$= \int_0^2 \left(x - \frac{4}{3}\right)^2 \frac{1}{2}x\,dx = \int_0^2 \left(x^2 - 2\,\frac{4}{3}x + \left(\frac{4}{3}\right)^2\right)\frac{1}{2}x\,dx$$

$$= \int_0^2 \frac{x^3}{2} - \frac{4x^2}{3} + \frac{16}{9}\,\frac{1}{2}x\,dx$$

$$= \frac{x^4}{8} - \frac{4x^3}{9} + \frac{4}{9}x^2\Big|_0^2 = \frac{2^4}{8} - \frac{4\,2^3}{9} + \frac{4\,2^2}{9}$$

$$= 2 - \frac{32}{9} + \frac{16}{9} = 2 - \frac{16}{9} = \boxed{\frac{2}{9}}$$

2)



$$E[k] = \sum_{k=1}^{6} k f(k) = \sum_{k=1}^{6} k\,\frac{1}{6} = 1\frac{1}{6} + 2\frac{1}{6} + 3\frac{1}{6} + 4\frac{1}{6} + 5\frac{1}{6} + 6\frac{1}{6}$$

$$= \frac{1}{6}(1+2+3+4+5+6) = \frac{1}{6}\,21 = \boxed{3.5}$$

$$Var[k] = \sum_{k=1}^{6} (k - E[k])^2 f(k) = \sum_{k=1}^{6} (k - 3.5)^2\,\frac{1}{6}$$

$$= \frac{(1-3.5)^2}{6} + \frac{(2-3.5)^2}{6} + \dots + \frac{(6-3.5)^2}{6}$$

$$= \frac{1}{6}\left((1-3.5)^2 + \dots + (6-3.5)^2\right)$$

$$= \frac{1}{6}(6.25 + 2.25 + 0.25 + 0.25 + 2.25 + 6.25)$$

$$= \frac{1}{6}(17.5) = \boxed{2.92}$$

3) $f(x) = k \qquad 0 < x < 2\pi$

$$A = 1 = k\,2\pi \implies k = \frac{1}{2\pi}$$

$$E[x] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{2\pi} kx\,dx = \frac{1}{4\pi}x^2\Big|_0^{2\pi} = \frac{(2\pi)^2}{4\pi}$$

$$= \frac{\cancel{4}\pi^{\cancel{2}}}{\cancel{4}\cancel{\pi}} = \boxed{\pi}$$

$$Var[x] = \int_{-\infty}^{\infty} (x - E(x))^2 f(x)\,dx = \int_0^{2\pi} (x - \pi)^2\,\frac{1}{2\pi}\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (x^2 - 2\pi x + \pi^2)\,dx = \frac{1}{2\pi}\left[\frac{x^3}{3} - \pi x^2 + \pi^2 x\Big|_0^{2\pi}\right.$$

$$= \frac{1}{\cancel{2\pi}}\left[\frac{(2\pi)^{\cancel{3}\,2}}{3} - \pi(2\pi)^{\cancel{2}} + \pi^2\cancel{(2\pi)}\right]$$

$$= \frac{4\pi^2}{3} - 2\pi^2 + \pi^2 = \pi^2\left(\frac{4}{3} - 2 + 1\right)$$

$$= \pi^2\left(\frac{4}{3} - 1\right) = \boxed{\pi^2/3}$$