pde.separation PDE solution by separation of variables

We are now ready to learn one of the most important techniques for solving PDEs: separation of variables. It applies only to linear PDEs since it will require the principle of superposition. Not all linear PDEs yield to this solution technique, but several that are important do.

separation of variables
linear

The technique includes the following steps.

assume a product solution Assume the solution can be written as a product solution u_p: the product of functions of each independent variable.

product solution

separate PDE Substitute u_p into the PDE and rearrange such that at least one side of the equation has functions of a single independent variabe. If this is possible, the PDE is called separable.

separable PDEs

set equal to a constant Each side of the equation depends on different independent variables; therefore, they must each equal the same constant, often called $-\lambda$.

repeat separation, as needed If there are more than two independent variables, there will be an ODE in the separated variable and a PDE (with one fewer variables) in the other independent variables. Attempt to separate the PDE until only ODEs remain.

solve each boundary value problem Solve each boundary value problem ODE, ignoring the initial conditions for now.

solve the time variable ODE Solve for the general solution of the time variable ODE, sans initial conditions.

construct the product solution Multiply the solution in each variable to construct the product solution u_p. If the boundary value problems were sturm-liouville, the product solution is a family of eigenfunctions from which any function can be constructed via a generalized fourier series.

eigenfunctions

apply the initial condition The product solutions individually usually do not meet the initial condition. However, a generalized fourier series of them nearly always does. Superposition tells us a linear combination of solutions to the PDE and boundary conditions is also a solution; the unique series that also satisfies the initial condition is the unique solution to the entire problem.

superposition

Example pde.separation−1 **re: 1D diffusion equation**

Consider the one-dimensional diffusion equation PDE[a]

$$\partial_t u(t,x) = k\partial_{xx}^2 u(t,x) \tag{1}$$

with real constant k, with dirichlet boundary conditions on inverval $x \in [0, L]$

$$u(t,0) = 0 \tag{2a}$$
$$u(t,L) = 0, \tag{2b}$$

change to neumann bc

$$\frac{\partial}{\partial x} u(t,x)\Big|_{x=0} = 0$$

$$\frac{\partial}{\partial x} u(t,x)\Big|_{x=L} = 0$$

and with initial condition

$$u(0,x) = f(x), \tag{3}$$

where f is some piecewise continuous function on $[0, L]$.

a. For more on the diffusion or heat equation, see Haberman, (Haberman, Applied Partial Differential Equations with Fourier Series and Boundary Value Problems (Classic Version), § 2.3) Kreyszig, (Kreyszig, Advanced Engineering Mathematics, § 12.5) and Strauss. (Strauss, Partial Differential Equations: An Introduction, § 2.3)

Assume a product solution

First, we assume a product solution of the form $u_p(t,x) = T(t)X(x)$ where T and X are unknown functions on $t > 0$ and $x \in [0, L]$.

Separate PDE

Second, we substitute the product solution into Eq. 1 and separate variables:

$$T'X = kTX'' \Rightarrow \tag{4}$$
$$\frac{T'}{kT} = \frac{X''}{X}. \tag{5}$$

So it is separable! Note that we chose to group k with T, which was arbitrary but conventional.

Set equal to a constant

Since these two sides depend on different independent variables (t and x), they must equal the same constant we call $-\lambda$, so we have two ODEs:

$$\frac{T'}{kT} = -\lambda \quad \Rightarrow T' + \lambda k T = 0 \tag{6}$$
$$\frac{X''}{X} = -\lambda \quad \Rightarrow X'' + \lambda X = 0. \tag{7}$$

Solve the boundary value problem

The latter of these equations with the boundary conditions (2) is precisely the same sturm-liouville boundary value problem from Example pde.sturm-1, which had eigenfunctions

$$X_n(x) = \sin\left(\sqrt{\lambda_n}x\right) \tag{8a}$$
$$= \sin\left(\frac{n\pi}{L}x\right) \tag{8b}$$

$$X_n(x) = \cos\left(\frac{n\pi}{L}x\right)$$

with corresponding (positive) eigenvalues

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2. \tag{9}$$

Solve the time variable ODE

The time variable ODE is homogeneous and has the familiar general solution

$$T(t) = ce^{-k\lambda t} \tag{10}$$

with real constant c. However, the boundary value problem restricted values of λ to λ_n, so

$$T_n(t) = ce^{-k(n\pi/L)^2 t}. \tag{11}$$

Construct the product solution

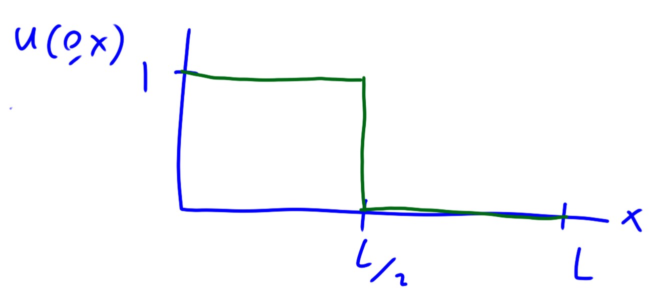
The product solution is

$$u_p(t,x) = T_n(t)X_n(x)$$
$$= ce^{-k(n\pi/L)^2 t}\sin\left(\frac{n\pi}{L}x\right). \tag{12}$$

$$u(t,x) = T_n(t)X_n(x)$$
$$= ce^{-K(n\pi/L)^2 t}\cos\left(\frac{n\pi}{L}x\right)$$

This is a family of solutions that each satisfy only exotically specific initial conditions.

given $f(x) = \begin{cases} 1 & x < \frac{L}{2} \\ 0 & \text{otherwise} \end{cases}$



Apply the initial condition

The initial condition is $u(0,x) = f(x)$. The eigenfunctions of the boundary value problem form a fourier series that satisfies the initial condition on the interval $[0, L]$ if we extend f to be periodic and odd over x (Kreyszig, Advanced Engineering Mathematics, p. 550); we call the extension f^*. The odd series synthesis can be written

$$f^*(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) \tag{13}$$

$$f^*(x) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right)$$

where the fourier analysis gives

$$b_n = \frac{2}{L}\int_0^L f^*(\chi)\sin\left(\frac{n\pi}{L}\chi\right). \tag{14}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx$$

So the complete solution is

$$u(t,x) = \sum_{n=1}^{\infty} b_n e^{-k(n\pi/L)^2 t}\sin\left(\frac{n\pi}{L}x\right). \tag{15}$$

$$u(t,x) = \sum_{n=1}^{\infty} a_n e^{-K\left(\frac{n\pi}{L}\right)^2 t}\cos\left(\frac{n\pi}{L}x\right)$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx$$

$$= \frac{2}{L}\int_0^{L/2}\cos\left(\frac{n\pi}{L}x\right)dx$$

$$= \frac{2}{\cancel{L}}\frac{\cancel{L}}{n\pi}\sin\left(\frac{n\pi}{L}x\right)\Big|_0^{L/2}$$

$$= \frac{2}{n\pi}\left(\sin\left(\frac{n\pi}{\cancel{L}}\frac{\cancel{L}}{2}\right) - \cancel{\sin\left(\frac{n\pi}{L}0\right)}\right)$$

$$a_n = \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

n	a_n
0	0
1	$\frac{2}{n\pi}$

$$u(t,x) = \sum_{n=1}^{\infty}\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)e^{-K\left(\frac{\pi n}{L}\right)^2 t}\cos\left(\frac{n\pi}{L}x\right)$$

a_n, b_n, a_0

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx = \frac{2}{L}\int_0^{L/2} 1\,dx$$

$$= \frac{2}{L}x\Big|_0^{L/2} = \frac{2}{L}\frac{L}{2} - \frac{2}{L}0 = 1$$

Notice this satisfies the PDE, the boundary conditions, and the initial condition!

Plotting solutions

If we want to plot solutions, we need to specify an initial condition $u(0,x) = f^*(x)$ over $[0, L]$. We can choose anything piecewise continuous, but for simplicity let's let

$$f(x) = 1. \qquad (x \in [0, L])$$

The odd periodic extension is an odd square wave. The integral (14) gives

$$b_n = \frac{4}{n\pi}(1 - \cos(n\pi))$$
$$= \begin{cases} 0 & n \text{ even} \\ \frac{4}{n\pi} & n \text{ odd.} \end{cases} \tag{16}$$

$x = [0 \ \dots \ L]$

$$n = \begin{bmatrix} 1 \\ 3 \\ \vdots \\ N \end{bmatrix} \qquad \frac{n\pi}{L}x = \begin{bmatrix} 0 & \dots & \pi \\ 0 & \dots & 3\pi \\ \vdots & \ddots & \vdots \\ 0 & \dots & N\pi \end{bmatrix}$$

Now we can write the solution as

$$u(t,x) = \sum_{n=1,\, n \text{ odd}}^{\infty}\frac{4}{n\pi}e^{-k(n\pi/L)^2 t}\sin\left(\frac{n\pi}{L}x\right). \tag{17}$$

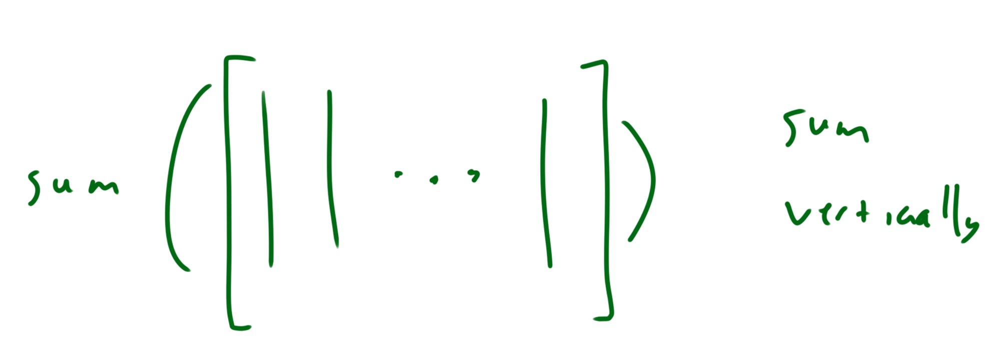
Plotting in Python

First, load some Python packages.

```
import numpy as np
import matplotlib.pyplot as plt
from IPython.display import display, Markdown, Latex
```

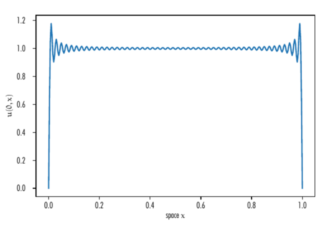
Set $k = L = 1$ and sum values for the first N terms of the solution.

```
L = 1
k = 1
N = 100
x = np.linspace(0,L,300)
t = np.linspace(0,2*(L/np.pi)**2,100)
u_n = np.zeros([len(t),len(x)])
for n in range(N):
  n = n+1 # because index starts at 0
  if n % 2 == 0: # even
    pass # already initialized to zeros
  else: # odd
    u_n += 4/(n*np.pi)*np.outer(
      np.exp(-k*(n*np.pi/L)**2*t),
      np.sin(n*np.pi/L*x)
    )
```



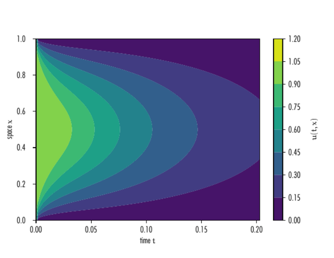
Let's first plot the initial condition.

```
p = plt.figure();
plt.plot(x,u_n[0,:]);
plt.xlabel('space $x$')
plt.ylabel('$u(0,x)$')
plt.show()
```



Now we plot the entire response.

```
p = plt.figure();
plt.contourf(t,x,u_n.T)
c = plt.colorbar()
c.set_label('$u(t,x)')
plt.xlabel('time $t$')
plt.ylabel('space $x$')
plt.show()
```



We see the diffusive action proceeds as we expected.

. Python code in this section was generated from a Jupyter notebook named pde_separation_example_01.ipynb with a python3 kernel.