pde.sturm Sturm−liouville problems

Before we introduce an important solution method for PDEs in Lec. pde.separation, we consider an ordinary differential equation that will arise in that method when dealing with a single spatial dimension x: the sturm-liouville (S-L) differential equation. Let p, q, σ be functions of x on open interval (a, b). Let X be the dependent variable and λ constant. The regular S-L problem is the S-L ODE[5]

sturm−liouville (S−L) differential equation

regular S−L problem

$$\frac{d}{dx}(pX') + qX + \lambda\sigma X = 0 \quad (1a)$$

with boundary conditions

$$\beta_1 X(a) + \beta_2 X'(a) = 0 \quad (2a)$$
$$\beta_3 X(b) + \beta_4 X'(b) = 0 \quad (2b)$$

with coefficients $\beta_i \in \mathbb{R}$. This is a type of boundary value problem.

boundary value problems

This problem has nontrivial solutions, called eigenfunctions $X_n(x)$ with $n \in \mathbb{Z}_+$, corresponding to specific values of $\lambda = \lambda_n$ called eigenvalues.[6] There are several important theorems proven about this (see Haberman[7]). Of greatest interest to us are that

eigenfunctions

eigenvalues

1. there exist an infinite number of eigenfunctions X_n (unique within a multiplicative constant),
2. there exists a unique corresponding real eigenvalue λ_n for each eigenfunction X_n,
3. the eigenvalues can be ordered as $\lambda_1 < \lambda_2 < \cdots$,
4. eigenfunction X_n has $n-1$ zeros on open interval (a, b),
5. the eigenfunctions X_n form an orthogonal basis with respect to weighting function σ such that any piecewise continuous function $f : [a, b] \to \mathbb{R}$ can be represented by a generalized fourier series on $[a, b]$.

This last theorem will be of particular interest in Lec. pde.separation.

5. For the S-L problem to be regular, it has the additional constraints that p, q, σ are continuous and $p, \sigma > 0$ on $[a, b]$. This is also sometimes called the sturm-liouville eigenvalue problem. See Haberman (Haberman, Applied Partial Differential Equations with Fourier Series and Boundary Value Problems (Classic Version), § 5.3) for the more general (non-regular) S-L problem and Haberman (ibidem, § 7.4) for the multi-dimensional analog.

6. These eigenvalues are closely related to, but distinct from, the "eigenvalues" that arise in systems of linear ODEs.

7. ibidem, § 5.3.

Types of boundary conditions

Boundary conditions of the sturm-liouville kind (2) have four sub-types:

dirichlet for just $\beta_2, \beta_4 = 0$,
neumann for just $\beta_1, \beta_3 = 0$,
robin for all $\beta_i \neq 0$, and
mixed if $\beta_1 = 0$, $\beta_3 \neq 0$; if $\beta_2 = 0$, $\beta_4 \neq 0$.

There are many problems that are not regular sturm-liouville problems. For instance, the right-hand sides of Eq. 2 are zero, making them homogeneous boundary conditions; however, these can also be nonzero. Another case is periodic boundary conditions:

homogeneous boundary conditions

periodic boundary conditions

$$X(a) = X(b) \quad (3a)$$
$$X'(a) = X'(b). \quad (3b)$$

Example pde.sturm−1 — **re: a sturm−liouville problem with dirichlet boundary conditions**

Consider the differential equation

$$X'' + \lambda X = 0 \quad (4)$$

with dirichlet boundary conditions on the boundary of the interval $[0, L]$

$$X(0) = 0 \quad \text{and} \quad X(L) = 0. \quad (5)$$

Solve for the eigenvalues and eigenfunctions.

bc $X'(0) = 0$ $X'(L) = 0$

This is a sturm-liouville problem, so we know the eigenvalues are real. The well-known general solutions to the ODE is

$$X(x) = \begin{cases} k_1 + k_2 x & \lambda = 0 \\ k_1 e^{j\sqrt{\lambda}x} + k_2 e^{-j\sqrt{\lambda}x} & \text{otherwise} \end{cases} \quad (6)$$

$X(x)$ given

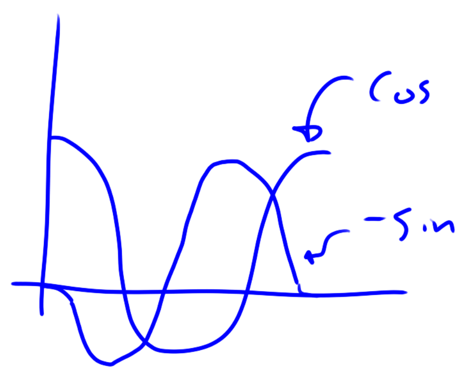
$X'(x)$

with real constants k_1, k_2. The solution must also satisfy the boundary conditions. Let's apply them to the case of $\lambda = 0$ first:

$$X(0) = 0 \Rightarrow k_1 + k_2(0) = 0 \Rightarrow k_1 = 0 \quad (7)$$
$$X(L) = 0 \Rightarrow k_1 + k_2(L) = 0 \Rightarrow k_2 = -k_1/L. \quad (8)$$

Together, these imply $k_1 = k_2 = 0$, which gives the trivial solution $X(x) = 0$, in which we aren't interested. We say, then, for nontrivial solutions $\lambda \neq 0$. Now let's check $\lambda < 0$. The solution becomes

$$X(x) = k_1 e^{-\sqrt{|\lambda|}x} + k_2 e^{\sqrt{|\lambda|}x} \quad (9)$$
$$= k_3 \cosh(\sqrt{|\lambda|}x) + k_4 \sinh(\sqrt{|\lambda|}x) \quad (10)$$



where k_3 and k_4 are real constants. Again applying the boundary conditions:

$$X(0) = 0 \Rightarrow k_3 \cosh(0) + k_4 \sinh(0) = 0 \Rightarrow k_3 + 0 = 0 \Rightarrow k_3 = 0$$
$$X(L) = 0 \Rightarrow 0 \cosh(\sqrt{|\lambda|}L) + k_4 \sinh(\sqrt{|\lambda|}L) = 0 \Rightarrow k_4 \sinh(\sqrt{|\lambda|}L) = 0.$$

However, $\sinh(\sqrt{|\lambda|}L) \neq 0$ for $L > 0$, so $k_4 = k_3 = 0$—again, the trivial solution. Now let's try $\lambda > 0$. The solution can be written

$$X(x) = k_5 \cos(\sqrt{\lambda}x) + k_6 \sin(\sqrt{\lambda}x). \quad (11)$$

$X'(x) = -k_5\sqrt{\lambda}\sin(\sqrt{\lambda}x) + k_6\cos(\sqrt{\lambda}x)\sqrt{\lambda}$

Applying the boundary conditions for this case:

$$X(0) = 0 \Rightarrow k_5 \cos(0) + k_6 \sin(0) = 0 \Rightarrow k_5 + 0 = 0 \Rightarrow k_5 = 0$$
$$X(L) = 0 \Rightarrow 0 \cos(\sqrt{\lambda}L) + k_6 \sin(\sqrt{\lambda}L) = 0 \Rightarrow k_6 \sin(\sqrt{\lambda}L) = 0.$$

$X'(0) = k_6 = 0$

$X'(L) = -k_5\sqrt{\lambda}\sin(\sqrt{\lambda}L) + ~~k_6\sqrt{\lambda}\cos(\sqrt{\lambda}L)~~ = 0$

Now, $\sin(\sqrt{\lambda}L) = 0$ for

$$\sqrt{\lambda}L = n\pi \Rightarrow$$
$$\lambda = \left(\frac{n\pi}{L}\right)^2. \quad (n \in \mathbb{Z}_+)$$

$-k_5\sqrt{\lambda}\sin(\sqrt{\lambda}L) = 0$

$\sin(\sqrt{\lambda}L) = 0$

$\sqrt{\lambda}L = n\pi$

$\lambda = \left(\frac{n\pi}{L}\right)^2$

Therefore, the only nontrivial solutions that satisfy both the ODE and the boundary conditions are the eigenfunctions

$$X_n(x) = \sin\left(\sqrt{\lambda_n}x\right) \quad (12a)$$
$$= \sin\left(\frac{n\pi}{L}x\right) \quad (12b)$$

with corresponding eigenvalues

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2. \quad (13)$$

$X_n(x) = \cos\left(\frac{n\pi}{L}x\right)$

Note that because $\lambda > 0$, λ_1 is the lowest eigenvalue.

Plotting the eigenfunctions

The following was generated from a Jupyter notebook with the following filename and kernel.

```
notebook filename: eigenfunctions_example_plot.ipynb
notebook kernel: python3
```

First, load some Python packages.

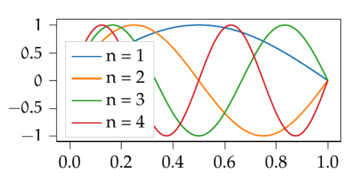
```
import numpy as np
import matplotlib.pyplot as plt
from IPython.display import display, Markdown, Latex
```

Set $L = 1$ and compute values for the first four eigenvalues `lambda_n` and eigenfunctions `X_n`.

```
L = 1
x = np.linspace(0,L,100)
n = np.linspace(1,4,4,dtype=int)
lambda_n = (n*np.pi/L)**2
X_n = np.zeros([len(n),len(x)])
for i,n_i in enumerate(n):
  X_n[i,:] = np.sin(np.sqrt(lambda_n[i])*x)
```

Plot the eigenfunctions.

```
for i,n_i in enumerate(n):
  plt.plot(
    x,X_n[i,:],
    linewidth=2,label='n = '+str(n_i)
  )
plt.legend()
plt.show() # display the plot
```



We see that the fourth of the S-L theorems appears true: $n-1$ zeros of X_n exist on the open interval $(0, 1)$.