$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \qquad X = [x_0, x_1, \dots, x_N]$$

$$x_{n+1} = x_n + dx$$

$$f'(x) = \lim_{dx \to 0} \frac{f(x+dx) - f(x)}{dx}$$ if dx is small

$$f'(x) \simeq \frac{f(x+dx) - f(x)}{dx} = \frac{f(x_{n+1}) - f(x_n)}{dx}$$ forward difference

$$f'(x_n) \simeq \frac{f(x_n) - f(x_{n-1})}{dx}$$ backwards difference

$$f''(x_n) \simeq \frac{f'(x_{n+1}) - f'(x_n)}{dx} = \frac{\frac{f(x_{n+1}) - f(x_n)}{dx} - \frac{f(x_n) - f(x_{n-1})}{dx}}{dx}$$

$$= \frac{f(x_{n+1}) - f(x_n) - f(x_n) + f(x_{n-1})}{dx^2}$$

$$= \frac{f(x_{n+1}) - 2f(x_n) + f(x_{n-1})}{dx^2}$$

$$\frac{\partial}{\partial t} u(t, x_n) = \alpha \frac{u(t, x_{n+1}) - 2u(t, x_n) + u(t, x_{n-1})}{dx^2}$$

$$\frac{d}{dt} u(t, x_n) = \alpha \frac{u(t, x_{n+1}) - 2u(t, x_n) + u(t, x_{n-1})}{dx^2}$$

IC $\quad u(0, x_n)$

BC $\quad u(t, x_0) = 0 \quad$ dirchlet

$u(t, x_0) = u(t, x_1)$

$$\frac{d}{dx} u(t, x_0) = \frac{u(t, x_1) - u(t, x_0)}{dx} = \frac{0}{dx} = 0$$

neumann

heat(u, t)

$$\frac{\alpha}{dx^2} \begin{bmatrix} 1 & -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdots \\ 0 & 0 & 1 & -2 & 1 & \cdots \\ 0 & \cdots & 0 & 1 & -2 & 1 \end{bmatrix} u$$

2nd order forward

$$f''(x_n) = \frac{f(x_{n+2}) - 2f(x_{n+1}) + f(x_n)}{dx^2}$$

2nd order backwards

$$f''(x_n) = \frac{f(x_n) - 2f(x_{n-1}) - f(x_{n-2})}{dx^2}$$

using neumann BC $\left.\frac{\partial u}{\partial x}\right|_{x = x_0, x_N} = 0$

$$\frac{du}{dt} = \frac{\alpha}{dx^2} \begin{bmatrix} 0 & \cdots & & & & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & 0 \cdots & 0 \\ & & \ddots & & & \\ 0 & \cdots & 0 & 1 & -2 & 1 \\ 0 & \cdots & & & & 0 \end{bmatrix} \begin{bmatrix} u(t, x_0) \\ u(t, x_1) \\ u(t, x_2) \\ \vdots \\ u(t, x_{N-1}) \\ u(t, x_N) \end{bmatrix}$$

$$u(t, x_0) = u(t, x_1)$$

with $\begin{bmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 \cdots & 0 \\ & & \ddots & & \\ 0 & \cdots & 0 & 1 & -2 \end{bmatrix}$

$u(t, x_{-1}) = 0$

$u(t, x_{N+1}) = 0$

neumann BC

$$\left.\frac{\partial u}{\partial x}\right|_{x = x_{0,N}} = 0$$

at each time t_n

$u(t_n, x_0) = u(t_n, x_1)$

$$A = \begin{bmatrix} 1 & -2 & 1 & 0 & \cdots & & 0 \\ 1 & -2 & 1 & 0 & \cdots & & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & -2 & 1 & 0 \\ 0 & \cdots & & 0 & 1 & -2 & 1 \\ 0 & \cdots & & 0 & 1 & -2 & 1 \end{bmatrix}$$

Time varying BC

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0 \qquad u(t, x_0) = u(t, x_1)$$

$$u(t, x_N) = \sin\left(\frac{8\pi}{0.5} t\right)$$

$$= \sin(16\pi t)$$