Laplace Transforms

Keyszig Ch 6

$$\mathcal{L}(f(t)) = F(s) = \int_{0}^{\infty} e^{-st} f(t)\, dt$$

$$\hat{f}(w) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i t w}\, dt \qquad \text{Fourier Tranform}$$

Inverse Laplace Transform

$$\mathcal{L}^{-1}(F(s)) = f(t) = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{\gamma - iT}^{\gamma + iT} e^{st} F(s)\, ds$$

$$= \lim_{k\to\infty} \frac{(-1)^k}{k!} \left(\frac{k}{t}\right)^{k+1} F^{(k)}\left(\frac{k}{t}\right)$$

Use table for inverse

Linear

$$\mathcal{L}(af(t) + bg(t)) = a\mathcal{L}(f(t)) + b\mathcal{L}(g(t))$$

Derivatives

$$\mathcal{L}(f'(t)) = s\mathcal{L}(f(t)) - f(0)$$

$$\mathcal{L}(f''(t)) = s^2\mathcal{L}(f(t)) - sf(0) - f'(0)$$

Dirac Delta Function

$$\delta(t) = \lim_{k\to 0} \begin{cases} 1/k & 0 \le t \le k \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = \frac{1}{k}k = 1$$

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{L}(\delta(t)) = 1$$

Solve

$x' + x = \delta(t) \qquad x(0) = 1$

$sF(s) - x(0) + F(s) = 1$

$sF(s) - 1 + F(s) = 1$

$sF(s) + F(s) = 2$

$F(s)(s+1) = 2$

$F(s) = \frac{2}{s+1} \qquad \mathcal{L}^{-1}\left(\frac{2}{s+1}\right) = 2\mathcal{L}^{-1}\left(\frac{1}{s+1}\right)$

$\boxed{x(t) = 2e^{-t}}$

Solve

$x' + \cos(t)x = \delta(t) \qquad x(0) = 1$

$sF(s) - x(0) + \frac{s}{s^2+1}F(s) = 1$

$sF(s) - 1 + \frac{s}{s^2+1}F(s) = 1$

$sF(s) + \frac{s}{s^2+1}F(s) = 2$

$F(s)\left(s + \frac{s}{s^2+1}\right) = 2$

$$F(s) = \frac{2}{s + \frac{s}{s^2+1}} \cdot \frac{s^2+1}{s^2+1} = \frac{2s^2+2}{s^3+s+s} = \frac{2s^2+2}{s^3+2s}$$

$$= \frac{2s^2+2}{s(s^2+2)} = \frac{2s^2+2}{s(s+i\sqrt{2})(s-i\sqrt{2})}$$

$s^2 + 2 = 0 \Rightarrow s^2 = -2 \Rightarrow s = \pm i\sqrt{2}$

$$\frac{2s^2+2}{s(s+i\sqrt{2})(s-i\sqrt{2})} = \frac{A}{s} + \frac{B}{s+i\sqrt{2}} + \frac{C}{s-i\sqrt{2}}$$

multiply by s

$$\frac{2s^2+2}{(s+i\sqrt{2})(s-i\sqrt{2})} = A + \frac{sB}{s+i\sqrt{2}} + \frac{sC}{s-i\sqrt{2}}$$

Set s to zero

$$\frac{2}{i\sqrt{2}(-i\sqrt{2})} = A = \frac{2}{2} = 1$$

Multiply by $s+i\sqrt{2}$

$$\frac{2s^2+2}{s(s-i\sqrt{2})} = (s+i\sqrt{2})\frac{1}{s} + B + (s+i\sqrt{2})\frac{C}{s-i\sqrt{2}}$$

$s + i\sqrt{2} = 0 \qquad s = -i\sqrt{2}$

$$\frac{2(-i\sqrt{2})^2+2}{-i\sqrt{2}(-i\sqrt{2}-i\sqrt{2})} = B = \frac{2(-2)+2}{-i\sqrt{2}(-2i\sqrt{2})} = \frac{-4+2}{-4}$$

$$= \frac{1}{2}$$

$$\frac{2s^2+2}{s(s+i\sqrt{2})(s-i\sqrt{2})} = \frac{1}{s} + \frac{1/2}{s+i\sqrt{2}} + \frac{C}{s-i\sqrt{2}}$$

Multiply by $s-i\sqrt{2}$

$$\frac{2s^2+2}{s(s+i\sqrt{2})} = (s-i\sqrt{2})\frac{1}{s} + (s-i\sqrt{2})\frac{1/2}{s+i\sqrt{2}} + C$$

$s - i\sqrt{2} = 0 \Rightarrow s = i\sqrt{2}$

$$\frac{2(i\sqrt{2})^2+2}{i\sqrt{2}(i\sqrt{2}+i\sqrt{2})} = C = \frac{-4+2}{i\sqrt{2}(2i\sqrt{2})} = \frac{-4+2}{-4}$$

$$= 1/2$$

$$F(s) = \frac{1}{s} + \frac{1/2}{s+i\sqrt{2}} + \frac{1/2}{s-i\sqrt{2}}$$

$$x = \mathcal{L}^{-1}(F(s)) = 1 + \frac{1}{2}e^{-i\sqrt{2}t} + \frac{1}{2}e^{i\sqrt{2}t}$$

$$= 1 + \frac{1}{2}(\cos(-\sqrt{2}t) + i\sin(-\sqrt{2}t))$$

$$+ \frac{1}{2}(\cos(\sqrt{2}t) + i\sin(\sqrt{2}t))$$

$\sin(-x) = -\sin(x)$

$\cos(-x) = \cos(x)$

$$= 1 + \frac{1}{2}(\cos(\sqrt{2}t) - i\sin(\sqrt{2}t) + \cos(\sqrt{2}t) + i\sin(\sqrt{2}t))$$

$$= 1 + \frac{1}{2}2\cos(\sqrt{2}t)$$

$\boxed{x = 1 + \cos(\sqrt{2}t)}$

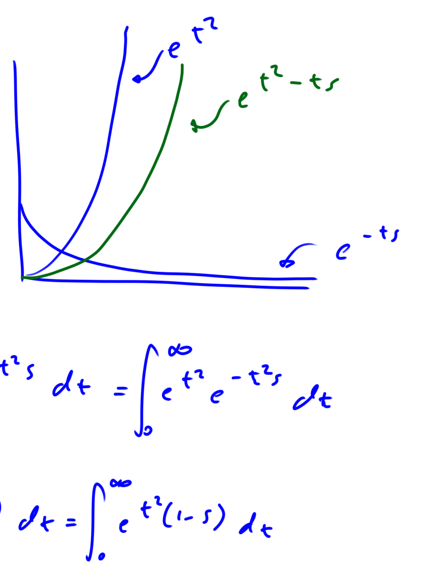
Limitations

$x' + x = u(t) \qquad x(-1) = 2 \qquad$ can't use Laplace Transform

$x' + x = e^{t^2} \qquad x(0) = 0$

$$\mathcal{L}(e^{t^2}) = \int_0^\infty e^{-ts} e^{t^2}\, dt = \int_0^\infty e^{t^2 - ts}\, dt = \infty$$

plot $e^{t^2 - ts}$



$$G(s) = \int_0^\infty f(t) e^{-t^2 s}\, dt = \int_0^\infty e^{t^2} e^{-t^2 s}\, dt$$

$$= \int_0^\infty e^{(t^2 - t^2 s)}\, dt = \int_0^\infty e^{t^2(1-s)}\, dt$$

$$= \frac{\sqrt{\pi}}{2\sqrt{s-1}} \qquad \text{if } |\arg(s-1)| \le \frac{\pi}{2}$$

Keyszig Problem 6.2.7

$y'' + 7y' + 12y = 21e^{3t} \qquad y(0) = 3.5 \qquad y'(0) = -10$

$$s^2F(s) - sy(0) - y'(0) + 7(sF(s) - y(0)) + 12F(s) = 21\frac{1}{s-3}$$

$$s^2F(s) - 3.5s + 10 + 7sF(s) - 24.5 + 12F(s) = 21\frac{1}{s-3}$$

$$s^2F(s) + 7sF(s) + 12F(s) = \frac{21}{s-3} + 3.5s - 10 + 24.5$$

$$F(s)(s^2 + 7s + 12) = \frac{21}{s-3} + 3.5s + 14.5$$

$$F(s) = \frac{\frac{21}{s-3} + 3.5s + 14.5}{s^2 + 7s + 12} \cdot \frac{s-3}{s-3}$$

$$= \frac{21 + 3.5s^2 - 10.5s + 14.5s - 43.5}{(s^2 + 7s + 12)(s-3)}$$

$$= \frac{3.5s^2 + 4s - 22.5}{(s+4)(s+3)(s-3)} = \frac{A}{s+4} + \frac{B}{s+3} + \frac{C}{s-3}$$

$s + 4 = 0 \Rightarrow s = -4$

$$\frac{3.5(-4)^2 + 4(-4) - 22.5}{(-4+3)(-4-3)} = A = \frac{17.5}{7} = 2.5$$

$s + 3 = 0 \Rightarrow s = -3$

$$\frac{3.5(-3)^2 + 4(-3) - 22.5}{(-3+4)(-3-3)} = B = \frac{-3}{-6} = 0.5$$

$s - 3 = 0 \Rightarrow s = 3$

$$\frac{3.5(3)^2 + 4(3) - 22.5}{(3+4)(3+3)} = C = \frac{21}{42} = 0.5$$

$$F(s) = \frac{2.5}{s+4} + \frac{0.5}{s+3} + \frac{0.5}{s-3}$$

$$\mathcal{L}(F(s)) = \boxed{y(t) = 2.5e^{-4t} + 0.5e^{-3t} + 0.5e^{3t}}$$