stab.routh Routh–Hurwitz criterion

There is no practical way to find the roots of a polynomial greater than degree four.[7] An implication of this is that we cannot practically solve (analytically) for the poles of a closed-loop transfer function with degree greater than four. Fortunately, numerical root finders can handle these higher-order systems with ease. However, there is a drawback to using numerical root finders to determine stability: design parameters, which show up in the coefficients of the denominator polynomial of a transfer function, must be assigned a specific value. A couple of mathematicians[8] in the late 19th century came up with a clever test—called the Routh-Hurwitz stability criterion[9]—for learning much about the stability of a system without computing its poles; moreover, the test yields an analytically tractable way to determine ranges over which design parameters yield stable closed-loop systems.

7. For the interested reader, see this stackexchange discussion.

8. Edward John Routh and Adolf Hurwitz were their names.

9. It is noteworthy that the criterion is based on the Routh-Hurwitz theorem.

An algorithm for applying the Routh-Hurwitz criterion

We consider an algorithm for this test. First, we address the "basic" algorithm and refer the reader to N. Nise (2015) for the two exceptions that arise when Column 1 has a zero or when an entire row is zero. You can teach this algorithm (including the exceptions) to a computer, as some have, but it is easy enough by-hand for many systems. Let the denominator of a closed-loop transfer function, with real coefficients a_i be

$$a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n,$$

where n a finite integer greater than or equal to the order of the numerator polynomial and $a_0 > 0$ (if it is not, make it so by multiplication by -1). Perform the following two steps.

Table routh.1: the general form of the Routh table. Empty cells are always zero.

	1	2	3	4	$\cdots$	
s^n	a_0	a_2	a_4	a_6	$\cdots$	$\cdots$
s^{n-1}	a_1	a_3	a_5	a_7	$\cdots$	$\cdots$
s^{n-2}	b_1	b_2	b_3	b_4	$\cdots$	$\cdots$
s^{n-3}	c_1	c_2	c_3	c_4	$\cdots$	$\cdots$
s^{n-4}	d_1	d_2	d_3	d_4	$\cdots$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		
s^2	e_1	e_2				
s^1	f_1					
s^0	g_1					

First, construct a Routh table. The procedure is to fill in the general form of the Routh table, shown in Table routh.1, with the definitions:

$$\begin{aligned}
b_1 &= -\frac{1}{a_1}\begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix}, & b_2 &= -\frac{1}{a_1}\begin{vmatrix} a_0 & a_4 \\ a_1 & a_5 \end{vmatrix}, & b_3 &= -\frac{1}{a_1}\begin{vmatrix} a_0 & a_6 \\ a_1 & a_7 \end{vmatrix}, & \cdots \\
c_1 &= -\frac{1}{b_1}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_2 \end{vmatrix}, & c_2 &= -\frac{1}{b_1}\begin{vmatrix} a_1 & a_5 \\ b_1 & b_3 \end{vmatrix}, & c_3 &= -\frac{1}{b_1}\begin{vmatrix} a_1 & a_7 \\ b_1 & b_4 \end{vmatrix}, & \cdots \\
d_1 &= -\frac{1}{c_1}\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}, & d_2 &= -\frac{1}{c_1}\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}, & d_3 &= -\frac{1}{c_1}\begin{vmatrix} b_1 & b_4 \\ c_1 & c_4 \end{vmatrix}, & \cdots \\
&\vdots & &\vdots & &\vdots \\
g_1 &= -\frac{1}{f_1}\begin{vmatrix} e_1 & e_2 \\ f_1 & 0 \end{vmatrix}, & g_2 &= -\frac{1}{f_1}\begin{vmatrix} e_1 & 0 \\ f_1 & 0 \end{vmatrix}, & g_3 &= -\frac{1}{f_1}\begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}.
\end{aligned} \tag{1}$$

Note the pattern that emerges in Equation 1. The number of rows and potentially nonzero columns are $n+1$ and $\lceil (n+1)/2 \rceil$. Potentially nonzero values hug Column 1. Descending rows, the number of potentially nonzero coefficients decreases.

The second step is to interpret the Routh table. For the basic Routh table, no poles lie on the imaginary axis (which excludes marginal stability), so interpretation is simple: the number of sign changes in Column 1 is equal to the number of poles in the right half-plane—and all others are in the left half-plane. Therefore, the system is strictly stable if its Routh array is of the basic type and has no sign changes in Column 1.

Example stab.routh−1 re: Basic Routh table with an unknown parameter

Given the closed-loop transfer function

$$\frac{s+7}{s^3+3s^2+s+k} \tag{2}$$

where k is a design parameter, using the Routh-Hurwitz criterion, find the range of k for which the closed-loop system is stable.

Let's build the Routh table in Table routh.2. The lower entries were computed from Equation 1 (n.b. we knew $b_2 = 0$, but compute it for demonstrative purposes) as follows:

$$b_1 = -\frac{1}{a_1}\begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix} = -\frac{1}{\_}\begin{vmatrix} \_ & \_ \\ \_ & \_ \end{vmatrix} = \_\_\_\_\_\_,$$

$$b_2 = -\frac{1}{a_1}\begin{vmatrix} a_0 & a_4 \\ a_1 & a_5 \end{vmatrix} = -\frac{1}{\_}\begin{vmatrix} \_ & \_ \\ \_ & \_ \end{vmatrix} = \_, \text{ and}$$

$$c_1 = -\frac{1}{b_1}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_2 \end{vmatrix} = -\frac{1}{\_\_\_\_\_\_}\begin{vmatrix} \_ & \_ \\ \_\_\_\_\_\_ & \_ \end{vmatrix} = \_.$$

Table routh.2: Routh table for Example stab.routh-1.

	1	2	3
s^3	_	_	0
s^2	_	_	0
s^1	_	_	0
s^0	_	0	0

$\rightarrow$

	1	2	3
s^3	_	_	0
s^2	_	_	0
s^1	______	_	0
s^0	_	0	0

Now we must interpret the result. Since the first two entries in Column 1 are positive, the last two must be in order for the system stability. The conditions are:

$$\_\_\_\_\_\_ > 0 \Rightarrow \_\_\_\_\_ \text{ and}$$
$$k > \_.$$

Therefore, the range for stability is ________. Expressed as an interval, $k \in$ _____.