trans.exact Exact analytical trans response char of first– and second–order sys

First-order systems without zeros

A first-order system without zeros has a transient response characterized by a time-constant τ that appears in the general response as

$$\llcorner e^{-t/\tau} + \llcorner\!\!\lrcorner. \tag{1}$$

The transient exponential decays such that in three time constants 3τ only 5 % of the term remains; in 5τ, less than 1 %.
There is neither peak nor overshoot for this type of response. However, the rise time for these systems is found by solving the time-domain differential equation

$$\tau\dot{y}(t) + y(t) = ku(t) \tag{2}$$

with output variable y, input variable u, and real constant k. It is easily shown that the solution to Eq. 2 in Eq. 2 is, for a unit step input,

$$y(t) = k\left(1 - e^{-t/\tau}\right), \tag{3}$$

from which we discover that the steady-state value is

$$\begin{align}
y_{ss} &= \lim_{t\to\infty} y(t) \tag{4a}\\
&= k. \tag{4b}
\end{align}$$

The rise time is, by definition, the duration of the time interval $[t_1, t_2]$ such that

$$\begin{align}
y(t_1) &= 0.1 y_{ss} \text{ to} \tag{5a}\\
y(t_2) &= 0.9 y_{ss}. \tag{5b}
\end{align}$$

The first of these yields

$$\begin{align}
k\left(1 - e^{-t_1/\tau}\right) = 0.1k \Rightarrow \tag{6a}\\
t_1 &= -\tau \ln 0.9 \tag{6b}\\
&\approx 0.1054\tau. \tag{6c}
\end{align}$$

Solving in an analogous fashion, we find $t_2 \approx 2.3026\tau$. The interval, then, is $t_2 - t_1 = 2.1972\tau$.

Equation 7 first−order system rise time

Finally, the settling time can be derived in a fashion similar to the rise time.

Equation 8 first−order system settling time

Second-order systems without zeros

Second-order system transient responses are characterized by a natural (angular) frequency ω_n and damping ratio ζ. It is helpful to recall the complex-plane graphical representation of the pole-zero plot for a second-order system without zeros, as shown in Fig. exact.1. Following a procedure very similar to that for first-order systems, the following relationships can be derived.

The rise time T_r does not have an analytical solution in terms of ω_n and ζ. However, Fig. exact.2 shows numerical solutions for T_r scaled by ω_n for $\zeta \in (0, 1)$.

The peak time T_p has the following, simple expression

$$T_p = \frac{\pi}{\omega_d}, \tag{9}$$

where $\omega_d = \omega_n\sqrt{1-\zeta^2}$ is the damped natural frequency.

The percent overshoot %OS is related directly to

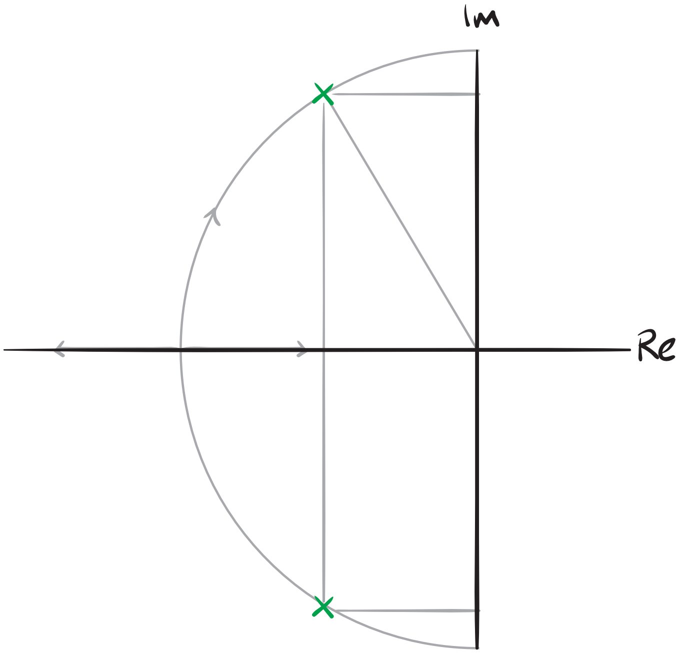

Figure exact.1: the relationship between the pole-zero plot of a second-order system with no zeros and ω_n and ζ.

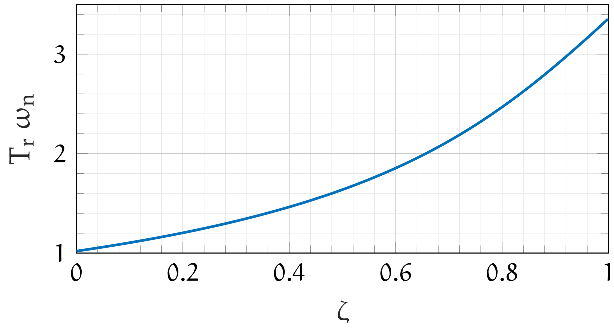

Figure exact.2: the relationship between rise time, natural frequency, and damping ratio.

ζ as follows

$$\%OS = 100 \exp \frac{-\zeta\pi}{\sqrt{1-\zeta^2}} \quad \Leftrightarrow \tag{10}$$

$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}. \tag{11}$$

Finally, the settling time T_s is expressed as

$$T_s = \frac{4}{\zeta\omega_n}. \tag{12}$$