rlocus.sketch Sketching the root locus

It is easy to get lost in the detailed rules of manual root locus construction. In the "old days" accurate root locus construction was critical, but now it is useful only for gaining intuition for how a given system will behave given its open-loop transfer function—which is extremely useful for design. If a detailed root locus is desired, we should use the computer tools of Lec. rlocus.comp.

We will construct a procedure for sketching a root locus from the following rules. In what follows phrases such as "has locus" are used to describe curves in the complex plane for which the root locus is defined. That is, everywhere in the complex plane for which the root locus is defined is said to "have locus." Note that some of the following rules only apply for $K > 0$.

R1. The root locus begins at open-loop poles, where $K = 0$, and approaches open-loop zeros, where $K \to \infty$. This was shown in Lec. rlocus.def to follow from the form of the closed-loop transfer function.

R2. The number of branches of the root locus is equal to the number of closed-loop poles. The number of closed-loop poles is equal to the number of open-loop poles or zeros, whichever is greater.

R3. The root locus is symmetric about the real-axis. This is due to the fact that poles can "leave" the real-axis only as conjugate pairs.

R4. On the real-axis, there is locus wherever an odd number of open-loop zeros and poles are on the real-axis, to the right—and no locus, elsewhere. This is a consequence of the phase criterion, Eq. 6. Recall, from Appendix A.01, the geometric evaluation of transfer functions. The phase criterion states that, for locus, $\angle KG(s)H(s)$ must always be π or its equivalent, so, for a test point ψ, the sum of the angles from each of the open-loop poles and zeros to ψ must be π or its equivalent, as can be illustrated in Fig. sketch.1. Due to the fact that every off-axis pair of open-loop poles or zeros contributes no net angle (because their angles are equally opposite), only poles and zeros on the real axis contribute to the phase of a given point on the real axis. When the point is to the right of every real-axis pole and zero, all the angle contributions are zero, and the phase criterion is not met. Moving the point leftward and it passes a pole or zero, the angle becomes $\pm\pi$, satisfying the angle criterion. Continuing leftward, each time it crosses a pole or zero, $\pm\pi$ is added, toggling satisfaction of the angle criterion.

R5. "Missing" poles and zeros are paired with infinite zeros and poles, asymptotically. An open-loop transfer function with a different number of poles and zeros is said to have "missing" poles or zeros. This is because the root locus begins at open-loop poles and approaches open-loop zeros—but what about systems with missing open-loop poles or zeros? For these situations, the root locus begins or ends at poles or zeros at infinity. For a system with more poles than zeros, which is quite common, some poles approach zeros at infinity, asymptotically. Conversely, for a system with more zeros than poles, which is uncommon and is called a non-causal system, some branches of the root locus begin asymptotically from poles at infinity. Asymptotes originate at a single real-axis intercept σ_a, which can be shown to be related to the finite poles p_i

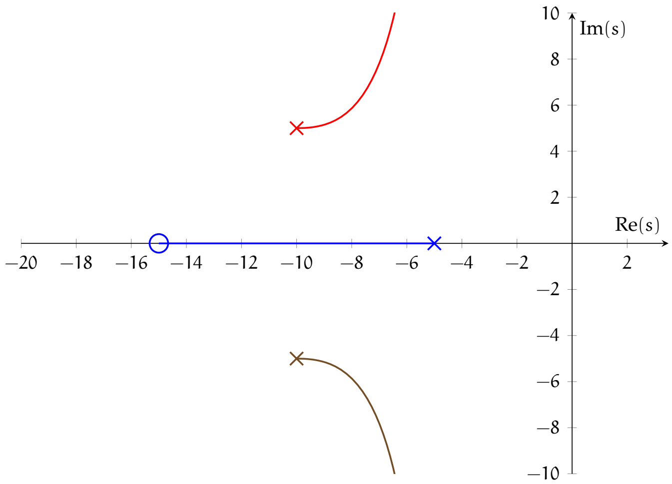

Figure sketch.1: a root locus example for illustrating the geometric interpretation of the phase criterion on the real axis.

and zeros z_j, with n_p and n_z the number of poles and zeros, as follows.

Equation 1 root locus asymptote real−axis intercept

Note that the imaginary parts of the poles and zeros cancel, so they needn't be considered. With $N \equiv n_p - n_z$, the number of asymptotes is $|N|$. Each is a ray that originates at σ_a, and all that remains undetermined is the angle of each ray, which can be shown to be as follows, for all $m \in \mathbb{Z}$.

Equation 2 root locus asymptote angles

Note that these repeat every $|N|$ consecutive values of θ_m.

Every root locus (with $K > 0$) will satisfy the rules above. They will help us construct sketches with the following procedure.

RL1. Sketch the open-loop poles and zeros. According to Rule R1, the root locus starts at the open-loop poles and ends at the open-loop zeros.

RL2. Sketch real-axis locus in accordance with Rule R4. Let's start with a win. Begin at the right of all real-axis poles and zeros (where there is never locus) and move leftward, toggling for each pole or zero, "no locus, locus, no locus, locus …."

RL3. If applicable, determine poles or zeros at infinity and draw asymptotes. Determine the number of finite poles n_p and finite zeros n_z. Compute $N = n_p - n_z$. If $N > 0$, there are $|N|$ zeros at infinity; if $N < 0$, there are $|N|$ poles at infinity; and if $N = 0$, there are neither poles nor zeros at infinity and the rest of this step should be skipped. Compute the asymptote real-axis intercept σ_a from Eq. 1. Compute $|N|$ asymptote angles $\theta_0, \theta_1, \cdots$ from Eq. 2. Sketch the asymptotes.

RL4. Finish the root locus sketch, respecting all rules. Typically, a qualitatively accurate sketch can now be constructed, which is our goal.

Example rlocus.sketch−1 **re: sketching the root locus**

Sketch the root locus for the open-loop transfer function

$$\frac{3(s+1)}{(s+3)(s+5)}.$$

Example rlocus.sketch−2 **re: sketching the root locus**

Sketch the root locus for the open-loop transfer function

$$\frac{53}{(s+5)(s^2+2s+2)}.$$