

ss.ssmodel State and output equations

1 The state \mathbf{x} , input \mathbf{u} , and output \mathbf{y} vectors interact through two equations:

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t) \quad (1a)$$

$$\mathbf{y} = \mathbf{g}(\mathbf{x}, \mathbf{u}, t) \quad (1b)$$

where \mathbf{f} and \mathbf{g} are vector-valued functions that depend on the system. Together, they comprise what is called a state-space model of a system. Let's not glide past these equations, which will be our dear friends for the rest of our analytic lives. The first equation (1a) is called the state equation. Given state and input vectors at a moment in time, its function \mathbf{f} describes, how the state is changing (i.e. $d\mathbf{x}/dt$). Clearly, the state equation is a vector differential equation, which is equivalent to a system of first-order differential equations.¹

state-space model

state equation

2 In accordance with the definition of a state-determined system from [Lecture ss.svar](#), given an initial condition $\mathbf{x}(t_0)$ and input \mathbf{u} , the state \mathbf{x} is determined for all $t \geq t_0$. The state-space model is precisely the "mathematical model" described in the definition of a state-determined system. Determining the state requires the solution—analytic or numerical—of the vector differential equation.

3 The second equation (1b) is algebraic. It expresses how the output \mathbf{y} can be constructed from the state \mathbf{x} and input \mathbf{u} . This means we must first solve the state equation (1a). Since the output \mathbf{y} is a vector of variables of interest, the output equation is constructed in two steps: (1) define the output variables and (2) write them in terms of the state variables x_i and input variables u_j .

4 Just because we know that, for a state-determined system, there exists a solution to [Equation 1a](#), doesn't mean we know how to

1. We'll learn how to solve such systems both analytically and numerically in later chapters.

find it. In general, $f : \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R} \rightarrow \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R} \rightarrow \mathbb{R}^m$ can be nonlinear functions.² We don't know how to solve most nonlinear state equations analytically. An additional complication can arise when, in addition to states and inputs, system parameters are themselves time-varying (note the explicit time t argument of f and g). Fortunately, often a linear model is sufficiently effective.³

5 A linear, time-invariant (LTI) system has state-space model

$$\frac{dx}{dt} = Ax + Bu \tag{2a}$$

$$y = Cx + Du \tag{2b}$$

where

- A is an $n \times n$ matrix that describes how the state x changes the state x ,
- B is an $n \times r$ matrix that describes how the input u changes the state x ,
- C is an $m \times n$ matrix that describes how the state x contributes to the output y , and
- D is an $m \times r$ matrix that describes how the input u contributes to the output y .

In the next two lectures, we will learn how to derive a state-space model—for linear systems, how to find A , B , C , and D —for a system from its linear graph. This is the link between the linear graph model and the state-space model.

2. Technically, since x and u are themselves functions, f and g are functionals.

3. A later lecture will describe the process of deriving a “linearized” model from a nonlinear one.

linear, time-invariant