emech.curves DC motor performance in steady–state

1 Brushed DC motor performance has several aspects, but most of them revolve around the so-called motor curve: for a given motor voltage, its steady-state speed versus a constant torque applied to the load. The test setup for drawing such a curve requires a calibrated, controllable torque source applied to the motor shaft. A brake is typically used. A voltage-controlled magnetic particle brake is ideal.[11]

2 We will gain a deep understanding of DC motor performance characteristics only by tarrying with this situation. Therefore, we begin by modeling it in Lecture emech.curves and analyzing its performance in Lecture emech.curves.

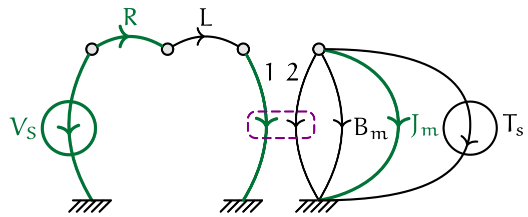

Figure curves.1: a linear graph model of the motor from Lecture emech.real in a test-configuration with a brake modeled by T_s.

motor curve
brake
magnetic particle brake

11. See, for instance here or here.

Modeling the test system

3 Including a torque source T_s on the load changes the model only slightly, as shown in Figure curves.1. Note that the mechanical subsystem is reduced to only the motor, since during such a test the load and bearings would be detrimental (it is a test for the motor, after all). Invariant are the normal tree, state variables, and most of the derivation of the state equations.

4 The input vector becomes

$$\mathbf{u} = \begin{bmatrix} V_s \\ T_s \end{bmatrix}. \quad (1)$$

The continuity equation for the inertia becomes $T_{J_m} = -T_2 - T_{B_m} - T_s$ (the torque specifically opposes motion, to which we assign the positive direction) and the state model's matrices B and D change, such that[12]

12. Here is the `rnd` file for use with statemint.stmartin.edu to derive the state-space model from the elemental, continuity, and compatibility equations.

$$A = \begin{bmatrix} -B_m/J_m & TF/J_m \\ -TF/L & -R/L \end{bmatrix}, \quad (2a)$$

$$B = \begin{bmatrix} 0 & -1/J_m \\ 1/L & 0 \end{bmatrix} \quad (2b)$$

$$C = \begin{bmatrix} 1 & -B_m & -TF & 0 & 1 & B_m & 0 & 0 & TF & 0 & 1 & 0 & 0 & 0 \\ 0 & TF & -R & 1 & 0 & 0 & R & 1 & 0 & 1 & 0 & -TF & 0 & 1 \end{bmatrix}^\top, \quad (2c)$$

$$D = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^\top. \quad (2d)$$

Steady-state performance analysis

Let's begin by defining the system parameters.

```
Kt_spec = 13.7; % oz-in/A ... torque constant from spec
Kv_spec = 10.2; % V/krpm ... voltage constant from spec
Tmax_spec = 2.82; % N-m ... max (stall) torque from spec
Omax_spec = 628; % rad/s ... max speed (no load) from spec
N_oz = 0.278013851; % N/oz
m_in = 0.0254; % m/in
Kt_si = Kt_spec*N_oz*m_in; % N-m/A
rads_krpm = 1e3*2*pi/60; % (rad/s)/krpm
Kv_si = Kv_spec/rads_krpm; % V/(rad/s)
Jm = 56.5e-6; % kg-m^2 ... inertia of rotor
Bm = 16.9e-6; % N-m/s^2 ... motor damping coef
R = 1.6; % Ohm ... armature resistance
L = 4.1e-3; % H ... armature inductance
TF = Kv_si; % N-m/A ... trans ratio/motor constant
```

Let's investigate what happens in steady-state $\overline{\mathbf{x}}$. The system is stationary when $\dot{\mathbf{x}} = \mathbf{0}$ and $\mathbf{u} = \overline{\mathbf{u}}$ (stationary),[13] so

13. A stationary input $\overline{\mathbf{u}}$ is required for a stationary state if the input has any effect on the state; that is, if B is nonzero.

$$\begin{aligned} \mathbf{0} &= A\overline{\mathbf{x}} + B\overline{\mathbf{u}} \Rightarrow \\ \overline{\mathbf{x}} &= -A^{-1}B\overline{\mathbf{u}}. \end{aligned} \quad (3)$$

Let's compute our steady-state solution for a constant voltage input $V_s(t) = \overline{V}$ and braking torque $T_s(t) = \overline{T}$. We use a symbolic approach to gain insight.

```
syms B_ J_ TF_ L_ R_ Vs_ Ts_ % using underscore for syms
```

```
a_ = [-B_/J_,TF_/J_;-TF_/L_,-R_/L_];
b_ = [0,-1/J_;1/L_,0];
u_ = [Vs_;Ts_];

M1_ = -inv(a_)*b_ % matrix -A^-1 B
den_ = TF_^2 + B_*R_; % common den
M2_ = M1_.*den_; % factor
xs_ = M1_*u_ % full ss sol
xs_2_ = M2_*u_; % naughty factorless ss sol
```

```
M1_ =

[ TF_/(TF_^2 + B_*R_), -R_/(TF_^2 + B_*R_)]
[  B_/(TF_^2 + B_*R_), TF_/(TF_^2 + B_*R_)]



xs_ =

 (TF_*Vs_)/(TF_^2 + B_*R_) - (R_*Ts_)/(TF_^2 + B_*R_)
 (B_*Vs_)/(TF_^2 + B_*R_) + (TF_*Ts_)/(TF_^2 + B_*R_)
```

```
eig(a_)
```

```
ans =

 -((B_^2*L_^2 - 2*B_*J_*L_*R_ + J_^2*R_^2 -
↪  4*J_*L_*TF_^2)^(1/2) + B_*L_ + J_*R_)/(2*J_*L_)
 -(B_*L_ - (B_^2*L_^2 - 2*B_*J_*L_*R_ + J_^2*R_^2 -
↪  4*J_*L_*TF_^2)^(1/2) + J_*R_)/(2*J_*L_)
```

A little more human-readably, using the fact that $\Omega_2 = \Omega_J$ and $i_1 = i_L$, and using bars to denote steady-state values,

$$\overline{\Omega_2} = \frac{1}{TF^2 + B_m R}(TF\overline{V_s} - R\overline{T_s}) \quad (4)$$

$$\overline{i_1} = \frac{1}{TF^2 + B_m R}(B\overline{V_s} + TF\overline{T_s}) \quad (5)$$

Let's focus on the first of these, the relationship between $\overline{\Omega_2}$ and $\overline{T_s}$. For given $\overline{V_s}$, there is a linearly decreasing relationship between $\overline{\Omega_2}$ and $\overline{T_s}$. This is precisely the motor curve. But it's one of a few curves plotted versus $\overline{T_s}$. Other common curves are current $\overline{i_1}$, mechanical braking power $\mathcal{P}_{brk} = \overline{T_s \Omega_s}$, and efficiency ε. The efficiency is defined as the ratio of the braking power to the voltage source power $\mathcal{P}_{src} = \overline{I_s V_s}$; i.e.

$$\varepsilon = \mathcal{P}_{brk} / \mathcal{P}_{src}. \qquad (6)$$

We already have expressions for $\overline{\Omega_2}$ and $\overline{i_1}$ in terms of $\overline{T_s}$, but we must still derive them for $\mathcal{P}_{brk}$ and ε. For $\mathcal{P}_{brk}$, we must express $\overline{\Omega_s}$ in terms of known quantities. From the linear graph, it is obvious that $\overline{\Omega_s} = \overline{\Omega_2}$. Therefore,

$$\mathcal{P}_{brk} = \overline{T_s \Omega_2}. \qquad (7)$$

Now for ε. We have the unknown source current $\overline{I_s}$. However, from the linear graph, it is obvious that $\overline{I_s} = \overline{i_1}$. Therefore,

$$\varepsilon = \frac{\overline{T_s \Omega_2}}{\overline{i_1 V_s}}. \qquad (8)$$

Let's compute these quantities for our parameters.

```
Vs = 60; % V ... max used, which is common
Tmax = TF/R*Vs; % N-m ... occurs when Omega_J = 0
Ts_a = linspace(0,Tmax,180); % N-m ... braking torques
O2_a = 1/(TF^2 + Bm*R)*(TF*Vs-R*Ts_a); % rad/s ... ss speed
i1_a = 1/(TF^2 + Bm*R)*(Bm*Vs+TF*Ts_a);
Pbrk_a = Ts_a.*O2_a; % W ... braking power
eff_a = Pbrk_a./(i1_a*Vs);
```

Now let's plot them! The output is shown in Figure curves.2.

There are some key quantities that can be read from the graph and found analytically. The most important are the maximum speed $\overline{\Omega_2}_{max}$, which occurs at zero torque, and maximum torque $\overline{T_s}_{max}$, which occurs at zero speed. Another is that the maximum mechanical power (output) occurs at $\overline{T_s}_{max}/2$. Finally, the maximum effieciency occurs at relatively low torque and high speed, which is typical for the

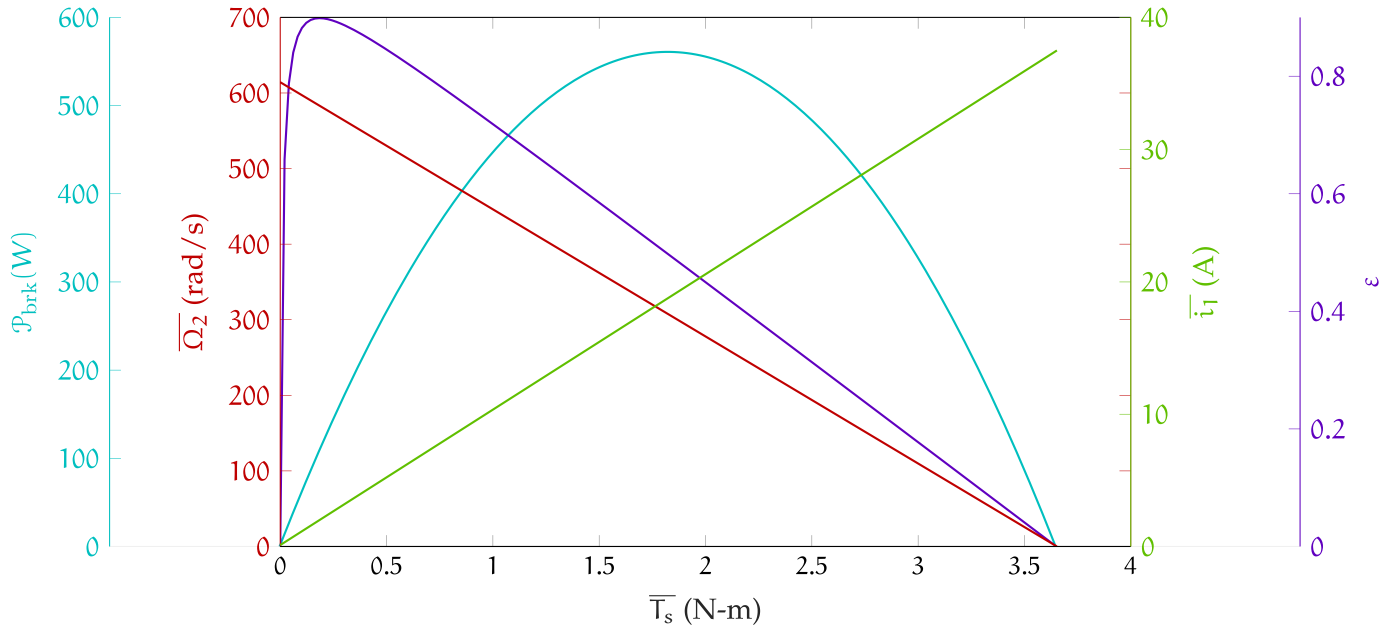

Figure curves.2: motor curves derived from the model.

following reason: the two energy-dissipative elements, the resistor and the damper, trade-off as being the dominant effect at the peak, and the resistor tends to dominate. That is, at high speed/voltage and low torque/current, the damper dominates dissipation; at low speed/voltage and high torque/current, the resistor dominates dissipation. It is very common for a motor's resistance to dominate the damping, as in our case.
Let's examine the maximum speed and torque.

```
Omax = O2_a(1) % rad/s ... occurs when T_s = 0
Tmax % N-m ... already computed and occurs when Omega_2 = 0
```

```
Omax =

  614.2479


Tmax =

    3.6526
```

Comparing these to the values given in the spec sheet, we see we're pretty good, but there's a bit of a discrepency in the max torque.

```
Omax_spec
Tmax_spec
disp(sprintf('percent error for speed: %0.3g',...
    (Omax-Omax_spec)/Omax_spec*100))
disp(sprintf('percent error for torque: %0.3g',...
    (Tmax-Tmax_spec)/Tmax_spec*100))
```

```
Omax_spec =

   628


Tmax_spec =

    2.8200

percent error for speed: -2.19
percent error for torque: 29.5
```

We should investigate further, but what we will find is that these values are fairly sensitive to TF, B, and R. In our case, it is likely that the given value for R is a bit low. It is given as 1.6 Ω, but it is probably closer to 2 Ω. However, the datasheet for this motor was not clear about whether the maximum speed and torque values were derived from a full motor curve fit or if they were the only points measured. The former is best for estimating dynamic model parameters like R and B, but the latter is occasionally sufficient.