ssresp.response   Solving for the state−space response

1   In this lecture, we solve the state equation for the state response $x(t)$ and substitute this into the output equation for the output response $y(t)$.

state response

output response

State response

2   The state equation can be solved by a synthesis of familiar techniques, as follows. First, we rearrange:

$$\frac{dx}{dt} - Ax = Bu. \tag{1}$$

An integrating factor would be clutch, but what should it be? It looks analogous to a scalar ODE that would use the natural exponential $\exp(-at)$ (for positive constant a), but we have a vector ODE. We need a matrix-version of the exponential. Recall that a series definition of the scalar exponential function $\exp : \mathbb{C} \to \mathbb{C}$ is

We define the matrix exponential $\exp : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n \times \mathbb{C}^n$ (we use the same symbol) to be, for $n \times n$ complex matrix Z,
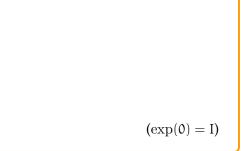
matrix exponential

$$\exp Z = \sum_{k=0}^{\infty} \frac{1}{k!} Z^k. \tag{2}$$

because why not? For the hell of it, let's see if the matrix exponential

$$\exp(-At) \tag{3}$$

works as an integrating factor, if for no other reason than it was constructed to be a sort of matrix-analog of $\exp(-at)$, which would work for the scalar case. Premultiplying (1) on both sides:

(exercise: prove $d\exp(-At)/dt = -\exp(-At)A$)

Rearranging and integrating over the interval $(0, t)$,

$(\exp(0) = I)$

This last expression can be solved for x, the state response solution. Before we do this, however, let's define the matrix function called the state transition matrix Φ to be the matrix-valued function

state response solution

state transition matrix

$$\Phi(t) = \exp(At), \qquad (4)$$

Substituting Φ and solving,

$$x = \Phi(t)x(0) + \Phi(t)\int_0^t \Phi(-\tau)Bu(\tau)d\tau \qquad (5a)$$

$$= \Phi(t)x(0) + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau. \qquad (5b)$$

Note that the first term of each version of Eq. 5 is the free response (due to initial conditions) and the second term is the forced response (due to inputs).

free response

forced response

State transition matrix

3   The state transition matrix Φ introduced in Eq. 4 wound up being a key aspect of the response, which is why we call it that. We used two of its properties (in matrix exponential form) during that derivation: the initial-value

initial−value property

$$\Phi(0) = I \qquad \text{(where I is the identity matrix)}$$

and the inverse

inverse property

$$\Phi^{-1}(t) = \Phi(-t). \qquad (6)$$

4   There is a third property that might be called
the bootstrapping property: for time intervals
Δt_i,

bootstrapping property

$$\Phi(\Delta t_1 + \Delta t_2 + \cdots) = \Phi(\Delta t_1)\Phi(\Delta t_2)\cdots. \qquad (7)$$

This allows one to compute the state transition
matrix[2] incrementally, from one previously
computed.

2. As is common, we refer to it as the "state transition matrix at a certain
time," but, technically, it's the image of the state transition matrix (which
is actually a matrix-valued function) at a certain time. It is good to
occasionally acknowledge the violence we do to math.

5   A final property we'll consider is the
special-case of a diagonal A with diagonal
elements $a_{11}, a_{22}, \cdots, a_{nn}$, which yields a
diagonal state transition matrix

diagonal property

6   The last property turns out to be quite
convenient for deriving Φ for a given system, as
we will see in Lec. ssresp.diag. For now, we
must rely on the definition of Φ from Eq. 4 and
the series definition of the matrix exponential
from Eq. 2. This requires us to derive the first
several terms of the series solution and attempt
to divine the corresponding scalar exponential
series, a rather tedious task. Other than to
familiarize ourselves with the definition
through exercises, we prefer the derivation
method of Lec. ssresp.diag.

deriving Φ

Output response

7   The output response $y(t)$ requires little
additional solution: assuming we have solved
for the state response $x(t)$, the output is given in

output response solution

the output equation Eq. 2b. Through direct
substitution, we find the output response
solution

$$\mathbf{y}(t) = C\mathbf{x}(t) + D\mathbf{u}(t) \tag{8a}$$

$$= C\Phi(t)\mathbf{x}(0) + C\int_0^t \Phi(t-\tau)B\mathbf{u}(\tau)d\tau + D\mathbf{u}(t). \tag{8b}$$