four.series Fourier series

1 Fourier series are mathematical series that can represent a periodic signal as a sum of sinusoids at different amplitudes and frequencies. They are useful for solving for the response of a system to periodic inputs. However, they are probably most important conceptually: they are our gateway to thinking of signals in the frequency domain—that is, as functions of frequency (not time). To represent a function as a Fourier series is to analyze it as a sum of sinusoids at different frequencies[1] ω_n and amplitudes a_n. Its frequency spectrum is the functional representation of amplitudes a_n versus frequency ω_n.

frequency domain

Fourier analysis

1. It's important to note that the symbol ω_n, in this context, is not the natural frequency, but a frequency indexed by integer n.

frequency spectrum

2 Let's begin with the definition.

Definition four.1: Fourier series: trigonometric form

The Fourier analysis of a periodic function $y(t)$ is, for $n \in \mathbb{N}_0$, period T, and angular frequency $\omega_n = 2\pi n/T$,

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} y(t)\cos(\omega_n t)dt \tag{1}$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} y(t)\sin(\omega_n t)dt. \tag{2}$$

The Fourier synthesis of a periodic function $y(t)$ with analysis components a_n and b_n corresponding to ω_n is

$$y(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\omega_n t) + b_n \sin(\omega_n t). \tag{3}$$

3 Let's consider the complex form of the Fourier series, which is analogous to Definition four.1. It may be helpful to review Euler's formula(s) – see Appendix com.euler.

Definition four.2: Fourier series: complex form

The Fourier analysis of a periodic function $y(t)$ is, for $n \in \mathbb{N}_0$, period T, and angular frequency $\omega_n = 2\pi n/T$,

$$c_{\pm n} = \frac{1}{T}\int_{-T/2}^{T/2} y(t)e^{-j\omega_n t}dt. \quad (4)$$

The Fourier synthesis of a periodic function $y(t)$ with analysis components c_n corresponding to ω_n is

$$y(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\omega_n t}. \quad (5)$$

4 We call the integer n a harmonic and the frequency associated with it, **harmonic**

$$\omega_n = 2\pi n/T, \quad (6)$$

the harmonic frequency. There is a special name for the first harmonic ($n = 1$): the fundamental frequency. It is called this because all other frequency components are integer multiples of it. **harmonic frequency** **fundamental frequency**

5 It is also possible to convert between the two representations above.

Definition four.3: Fourier series: converting between forms

The complex Fourier analysis of a periodic function $y(t)$ is, for $n \in \mathbb{N}_0$ and a_n and b_n as defined above,

$$c_{\pm n} = \frac{1}{2}\left(a_{|n|} \mp j b_{|n|}\right) \quad (7)$$

The sinusoidal Fourier analysis of a periodic function $y(t)$ is, for $n \in \mathbb{N}_0$ and c_n as defined above,

$$a_n = c_n + c_{-n} \text{ and} \quad (8)$$
$$b_n = j\left(c_n - c_{-n}\right). \quad (9)$$

6 The harmonic amplitude C_n is **harmonic amplitude**

$$C_n = \sqrt{a_n^2 + b_n^2} \quad (10)$$

$$= 2\sqrt{c_n c_{-n}}. \quad (11)$$

A magnitude line spectrum is a graph of the harmonic amplitudes as a function of the harmonic frequencies. The harmonic phase is

magnitude line spectrum

harmonic phase

$$\theta_n = -\arctan_2(b_n, a_n)$$

(see Appendix math.trig)

$$= \arctan_2(\operatorname{Im}(c_n), \operatorname{Re}(c_n)). \quad (12)$$

7 The illustration of Fig. series.1 shows how sinusoidal components sum to represent a square wave. A line spectrum is also shown.

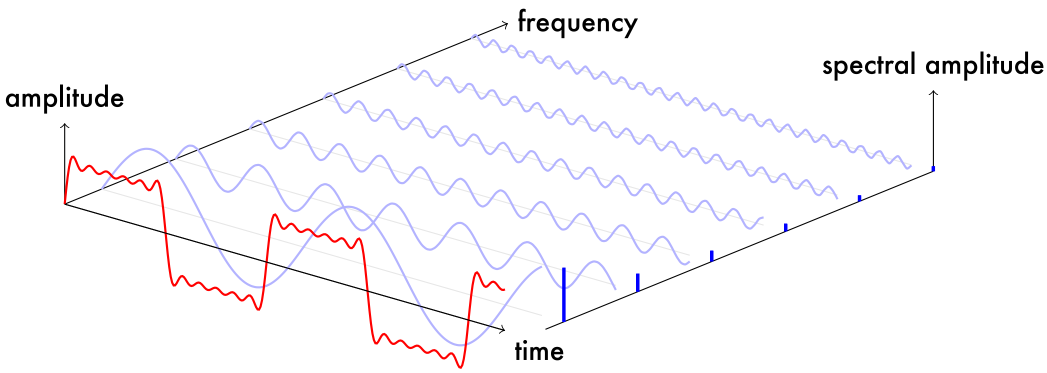

Figure series.1: a partial sum of Fourier components of a square wave shown through time and frequency. The spectral amplitude shows the amplitude of the corresponding Fourier component.

8 Let us compute the associated spectral components in the following example.

Example four.series–1

re: Fourier series analysis: line spectrum

Compute the first five harmonic amplitudes that represent the line spectrum for a square wave in the figure above.