lap.sol Solving io ODEs with Laplace

1 Laplace transforms provide a convenient method for solving input-output (io) ordinary differential equations (ODEs).

2 Consider a dynamic system described by the ________________—with t time, y the output, u the input, constant coefficients a_i, b_j, order n, and $m \leqslant n$ for $n \in \mathbb{N}_0$—as:

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1} u}{dt^{m-1}} + \cdots + b_1 \frac{du}{dt} + b_0 u. \quad (1)$$

Re-written in summation form,

$$\sum_{i=0}^{n} a_i y^{(i)}(t) = \sum_{j=0}^{m} b_j u^{(j)}(t), \quad (2)$$

where we use Lagrange's notation for derivatives, and where, ________________________, $a_n = 1$.

3 The Laplace transform $\mathcal{L}$ of Eq. 2 yields

$$\mathcal{L}\sum_{i=0}^{n} a_i y^{(i)}(t) = \mathcal{L}\sum_{j=0}^{m} b_j u^{(j)}(t) \quad \Rightarrow \quad (3a)$$

$$\sum_{i=0}^{n} a_i \mathcal{L}\left(y^{(i)}(t)\right) = \sum_{j=0}^{m} b_j \mathcal{L}\left(u^{(j)}(t)\right). \quad \text{(linearity)}$$

In the next move, we recursively apply the ________________________ property to yield the following

$$\sum_{i=0}^{n} a_i \left(s^i Y(s) + \underbrace{\sum_{k=1}^{i} s^{i-k} y^{(k-1)}(0)}_{I_i(s)} \right) = \sum_{j=0}^{m} b_j s^j U(s), \quad (4)$$

where terms in $I_i(s)$ arise from the __________ ______________. Splitting the left outer sum and solving for $Y(s)$,

$$\sum_{i=0}^{n} a_i s^i Y(s) + \sum_{i=0}^{n} a_i I_i(s) = \sum_{j=0}^{m} b_j s^j U(s) \quad \Rightarrow \tag{5a}$$

$$\sum_{i=0}^{n} a_i s^i Y(s) = \sum_{j=0}^{m} b_j s^j U(s) - \sum_{i=0}^{n} a_i I_i(s) \quad \Rightarrow \tag{5b}$$

$$Y(s) \sum_{i=0}^{n} a_i s^i = U(s) \sum_{j=0}^{m} b_j s^j - \sum_{i=0}^{n} a_i I_i(s) \quad \Rightarrow \tag{5c}$$

$$Y(s) = \underbrace{\frac{\sum_{j=0}^{m} b_j s^j}{\sum_{i=0}^{n} a_i s^i} U(s)}_{Y_{fo}(s)} + \underbrace{\frac{-\sum_{i=0}^{n} a_i I_i(s)}{\sum_{i=0}^{n} a_i s^i}}_{Y_{fr}(s)}. \tag{5d}$$

4 So we have derived the ____________ ____________________ $Y(s)$ in terms of the forced and free responses (still in the s-domain, of course)! For a solution in the time-domain, we must inverse Laplace transform:

forced
free

$$y(t) = \underbrace{(\mathcal{L}^{-1} Y_{fo})(t)}_{y_{fo}(t)} + \underbrace{(\mathcal{L}^{-1} Y_{fr})(t)}_{y_{fr}(t)}. \tag{6}$$

This is an important result!

Example lap.sol–1

Consider a system with step input $u(t) = 7u_s(t)$, output $y(t)$, and io ODE

$$\ddot{y} + 2\dot{y} + y = 2u. \tag{7}$$

Solve for the forced response $y_{fo}(t)$ with Laplace transforms.

5 From Eq. 6,

$$\begin{aligned} y_{fo}(t) &= (\mathcal{L}^{-1} Y_{fo})(t) \\ &= \mathcal{L}^{-1}\left(\frac{\sum_{j=0}^{m} b_j s^j}{\sum_{i=0}^{n} a_i s^i} U(s)\right) && \text{(Eq. 5d)} \\ &= \mathcal{L}^{-1}\left(\frac{2}{s^2 + 2s + 1} U(s)\right). && \text{(Eq. 7)} \end{aligned}$$

We can __________________________ $u(t)$ for

$U(s)$:

$$\begin{aligned} U(s) &= (\mathcal{L}u)(s) \\ &= 7(\mathcal{L}u_s)(s) \\ &= \frac{7}{s}, \end{aligned}$$

where the last equality follows from a transform easily found in Table lap.1.

6 Returning to the time response ______ ______,

$$\begin{aligned} y_{fo}(t) &= \mathcal{L}^{-1}\left(\frac{2}{s^2+2s+1}U(s)\right) \\ &= \mathcal{L}^{-1}\left(\frac{2}{s^2+2s+1}\cdot\frac{7}{s}\right). \end{aligned}$$

7 We can use Matlab's Symbolic Math toolbox function `partfrac` to perform the partial fraction expansion.

```
syms s 'complex'
Y = 2/(s^2 + 2*s + 1)*7/s;
Y_pf = partfrac(Y)
```

```
Y_pf =

14/s - 14/(s + 1)^2 - 14/(s + 1)
```

Or, a little nicer to look at,

$$Y(s) = 14\left(\frac{1}{s} - \frac{1}{(s+1)^2} - \frac{1}{s+1}\right).$$

Substituting this into our solution,

$$\begin{aligned} y_{fo}(t) &= 14\mathcal{L}^{-1}\left(\frac{1}{s} - \frac{1}{(s+1)^2} - \frac{1}{s+1}\right) && \text{(linearity)} \\ &= 14\left(\mathcal{L}^{-1}\frac{1}{s} - \mathcal{L}^{-1}\frac{1}{(s+1)^2} - \mathcal{L}^{-1}\frac{1}{s+1}\right) \\ &= 14\left(u_s(t) - te^{-t} - e^{-t}\right) && \text{(Table lap.1)} \\ &= 14\left(u_s(t) - (t+1)e^{-t}\right). \end{aligned}$$

So the forced response starts at 0 and decays ______ to a steady 14.