06.2 Difference equations

Many continuous dynamic systems can be described by a linear, constant-coefficient differential equation:

$$\alpha_n \frac{d^n y}{dt^n} + \alpha_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots + \alpha_1 \frac{dy}{dt} + \alpha_0 y = \\ = \beta_m \frac{d^m x}{dt^m} + \beta_{m-1} \frac{d^{m-1} x}{dt^{m-1}} + \ldots + \beta_1 \frac{dx}{dt} + \beta_0 x \tag{1}$$

where α_k and β_k are constants.
The corresponding discrete system is described by a difference equation that operates on the sequence of input values $x(n)$ to produce the output sequence $y(n)$. The difference equation has the form

$$a_0 y(n) + a_1 y(n-1) + \ldots + a_N y(n-N) = \\ = b_0 x(n) + b_1 x(n-1) + \ldots + b_M x(n-M) \tag{2}$$

for $n = 0, 1, 2, \ldots,$ where $x(n)$ is a sequence of periodically digitized values of the analog input signal, $y(n)$ is a sequence of values that determine the output signal, and a_k for $k = 0, 1, \ldots, N$ and b_k for $k = 0, 1, \ldots, M$ are constants.
This equation can also be written in summation form:

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k) \tag{3}$$

or, solving this for the current output sample $y(n)$,

$$y(n) = \frac{1}{a_0} \left[\sum_{k=0}^{M} b_k x(n-k) - \sum_{k=1}^{N} a_k y(n-k) \right] \tag{4}$$

Notice that the current output value $y(n)$ depends on previous values of y and on the previous and current values of the input x.
The problem of finding a discrete approximation of a continuous dynamic system

represented by the differential equation Eq. 1, then, is now just the problem of finding appropriate constants a_k and b_k in the difference equation such that its behavior approximates that of Eq. 1 with its constants α_k and β_k.

It turns out the best methods of approximation are derived not directly from the differential-difference equation relationship, but instead from the (implied) continuous-discrete transfer function relationship thereof. It is to the discrete transfer function that we therefore turn.