itself.found The foundations of mathematics

1 Mathematics has long been considered exemplary for establishing truth. Primarily, it uses a method that begins with axioms—unproven propositions that include undefined terms—and uses logical deduction to prove other propositions (theorems): to show that they are necessarily true if the axioms are.

axioms
deduction
proof
theorems

2 It may seem obvious that truth established in this way would always be relative to the truth of the axioms, but throughout history this footnote was often obscured by the "obvious" or "intuitive" universal truth of the axioms.[15] For instance, Euclid (Wikipedia. Euclid — Wikipedia, The Free Encyclopedia. `http://en.wikipedia.org/w/index.php?title=Euclid&oldid=923031048`. [Online; accessed 26-October-2019]. 2019) founded geometry—the study of mathematical objects traditionally considered to represent physical space, like points, lines, etc.—on axioms thought so solid that it was not until the early 19th century that Carl Friedrich Gauss (Wikipedia. Carl Friedrich Gauss — Wikipedia, The Free Encyclopedia. `http://en.wikipedia.org/w/index.php?title=Carl%20Friedrich%20Gauss&oldid=922692291`. [Online; accessed 26-October-2019]. 2019) and others recognized this was only one among many possible geometries (M. Kline. Mathematics: The Loss of Certainty. A Galaxy book. Oxford University Press, 1982. ISBN: 9780195030853. A detailed account of the "illogical" development of mathematics and an exposition of its therefore remarkable utility in describing the world.) resting on different axioms. Furthermore, Aristotle (Christopher Shields. ?Aristotle? inThe Stanford Encyclopedia of Philosophy: byeditorEdward N. Zalta. Winter 2016. Metaphysics Research Lab, Stanford University,

15. Throughout this section, for the history of mathematics I rely heavily on Kline. (M. Kline. Mathematics: The Loss of Certainty. A Galaxy book. Oxford University Press, 1982. ISBN: 9780195030853. A detailed account of the "illogical" development of mathematics and an exposition of its therefore remarkable utility in describing the world.)

Euclid
geometry
Gauss
Aristotle

2016) had acknowledged that reasoning must begin with undefined terms; however, even Euclid (presumably aware of Aristotle's work) seemed to forget this and provided definitions, obscuring the foundations of his work and starting mathematics on a path that for over 2,000 years would forget its own relativity (Kline, Mathematics: The Loss of Certainty, p. 101-2).

3 The foundations of Euclid were even shakier than its murky starting point: several unstated axioms were used in proofs and some proofs were otherwise erroneous. However, for two millennia, mathematics was seen as the field wherein truth could be established beyond doubt.

Algebra ex nihilo

4 Although not much work new geometry appeared during this period, the field of algebra (Wikipedia. Algebra — Wikipedia, The Free Encyclopedia. http://en.wikipedia.org/w/index.php?title=Algebra&oldid=920573802. [Online; accessed 26-October-2019]. 2019)—the study of manipulations of symbols standing for numbers in general—began with no axiomatic foundation whatsoever. The Greeks had a notion of rational numbers, ratios of natural numbers (positive integers), and it was known that many solutions to algebraic equations were irrational (could not be expressed as a ratio of integers). But these irrational numbers, like virtually everything else in algebra, were gradually accepted because they were so useful in solving practical problems (they could be approximated by rational numbers and this seemed good enough). The rules of basic arithmetic were accepted as applying to these and other forms of new numbers that arose in algebraic solutions: negative, imaginary, and

algebra

rational numbers
natural numbers
integers

irrational numbers

negative numbers
imaginary numbers

complex numbers.

complex numbers

The application of mathematics to science

5 During this time, mathematics was being applied to optics and astronomy. Sir Isaac Newton then built calculus upon algebra, applying it to what is now known as Newtonian mechanics, which was really more the product of Leonhard Euler (George Smith. ?Isaac Newton? inThe Stanford Encyclopedia of Philosophy: byeditorEdward N. Zalta. Fall 2008. Metaphysics Research Lab, Stanford University, 2008; Wikipedia. Leonhard Euler — Wikipedia, The Free Encyclopedia. http://en.wikipedia.org/w/index.php?title=Leonhard%20Euler&oldid=921824700. [Online; accessed 26-October-2019]. 2019). Calculus introduced its own dubious operations, but the success of mechanics in describing and predicting physical phenomena was astounding. Mathematics was hailed as the language of God (later, Nature).

optics
astronomy
calculus
Newtonian mechanics

The rigorization of mathematics

6 It was not until Gauss created non-Euclidean geometry, in which Euclid's were shown to be one of many possible axioms compatible with the world, and William Rowan Hamilton (Wikipedia. William Rowan Hamilton — Wikipedia, The Free Encyclopedia. http://en.wikipedia.org/w/index.php?title=William%20Rowan%20Hamilton&oldid=923163451. [Online; accessed 26-October-2019]. 2019) created quaternions (Wikipedia. Quaternion — Wikipedia, The Free Encyclopedia. http://en.wikipedia.org/w/index.php?title=Quaternion&oldid=920710557. [Online; accessed 26-October-2019]. 2019), a number system in which multiplication is noncommunicative, that it became apparent something was

non−Euclidean geometry
quaternions

fundamentally wrong with the way truth in mathematics had been understood. This started a period of rigorization in mathematics that set about axiomatizing and proving 19th century mathematics. This included the development of symbolic logic, which aided in the process of deductive reasoning.

rigorization

symbolic logic

7 An aspect of this rigorization is that mathematicians came to terms with the axioms that include undefined terms. For instance, a "point" might be such an undefined term in an axiom. A mathematical model is what we create when we attach these undefined terms to objects, which can be anything consistent with the axioms.[16] The system that results from proving theorems would then apply to anything "properly" described by the axioms. So two masses might be assigned "points" in a Euclidean geometric space, from which we could be confident that, for instance, the "distance" between these masses is the Euclidean norm of the line drawn between the points. It could be said, then, that a "point" in Euclidean geometry is implicitly defined by its axioms and theorems, and nothing else. That is, mathematical objects are not inherently tied to the physical objects to which we tend to apply them. Euclidean geometry is not the study of physical space, as it was long considered—it is the study of the objects implicitly defined by its axioms and theorems.

mathematical model

16. The branch of mathematics called model theory concerns itself with general types of models that can be made from a given formal system, like an axiomatic mathematical system. For more on model theory, see Hodges. (Wilfrid Hodges. ?Model Theory? inThe Stanford Encyclopedia of Philosophy: byeditorEdward N. Zalta. Fall 2018. Metaphysics Research Lab, Stanford University, 2018) It is noteworthy that the engineering/science use of the term "mathematical model" is only loosely a "model" in the sense of model theory.

implicit definition

The foundations of mathematics are built

8 The building of the modern foundations mathematics began with clear axioms, solid reasoning (with symbolic logic), and lofty yet seemingly attainable goals: prove theorems to support the already ubiquitous mathematical techniques in geometry, algebra, and calculus from axioms; furthermore, prove that these

axioms (and things they imply) do not contradict each other, i.e. are consistent, and that the axioms are not results of each other (one that can be derived from others is a theorem, not an axiom).

consistent

theorem

9 Set theory is a type of formal axiomatic system that all modern mathematics is expressed with, so set theory is often called the foundation of mathematics (Joan Bagaria. ?Set Theory? inThe Stanford Encyclopedia of Philosophy: byeditorEdward N. Zalta. Fall 2019. Metaphysics Research Lab, Stanford University, 2019). We will study the basics in ??. The primary objects in set theory are sets: informally, collections of mathematical objects. There is not just one a single set of axioms that is used as the foundation of all mathematics for reasons will review in a moment. However, the most popular set theory is Zermelo-Fraenkel set theory with the axiom of choice (ZFC). The axioms of ZF sans C are as follows. (ibidem)

Set theory

foundation

sets

ZFC set theory

extensionality If two sets A and B have the same elements, then they are equal.

empty set There exists a set, denoted by $\varnothing$ and called the empty set, which has no elements.

pair Given any sets A and B, there exists a set, denoted by $\{A, B\}$, which contains A and B as its only elements. In particular, there exists the set $\{A\}$ which has A as its only element.

power set For every set A there exists a set, denoted by $\mathcal{P}(A)$ and called the power set of A, whose elements are all the subsets of A.

union For every set A, there exists a set, denoted by $\bigcup A$ and called the union of A, whose elements are all the elements of the elements of A.

infinity There exists an infinite set. In particular, there exists a set Z that contains $\varnothing$ and such that if $A \in Z$, then $\bigcup\{A, \{A\}\} \in Z$.

separation For every set A and every given property, there is a set containing exactly the elements of A that have that property. A property is given by a formula φ of the first-order language of set theory. Thus, separation is not a single axiom but an axiom schema, that is, an infinite list of axioms, one for each formula φ.

replacement For every given definable function with domain a set A, there is a set whose elements are all the values of the function.

10 ZFC also has the axiom of choice. (Bagaria, ?Set Theory?)

choice For every set A of pairwise-disjoint non-empty sets, there exists a set that contains exactly one element from each set in A.

The foundations have cracks

11 The foundationalists' goal was to prove that some set of axioms from which all of mathematics can be derived is both consistent (contains no contradictions) and complete (every true statement is provable). The work of Kurt Gödel (Juliette Kennedy. ?Kurt Gödel? inThe Stanford Encyclopedia of Philosophy: byeditorEdward N. Zalta. Winter 2018. Metaphysics Research Lab, Stanford University, 2018) in the mid 20$^{\text{th}}$ century shattered this dream by proving in his first incompleteness theorem that any consistent formal system within which one can do some amount of basic arithmetic is incomplete! His argument is worth reviewing (see Raatikainen[17]), but at its heart is an undecidable statement like "This sentence is

first incompleteness theorem

incomplete

17. Panu Raatikainen. ?Gödel's Incompleteness Theorems? inThe Stanford Encyclopedia of Philosophy: byeditorEdward N. Zalta. Fall 2018. Metaphysics Research Lab, Stanford University, 2018. A through and contemporary description of Gödel's incompleteness theorems, which have significant implications for the foundations and function of mathematics and mathematical truth.

undecidable

unprovable.” Let U stand for this statement. If it is true it is unprovable. If it is provable it is false. Therefore, it is true iff it is provable. Then he shows that if a statement A that essentially says “arithmetic is consistent” is provable, then so is the undecidable statement U. But if U is to be consistent, it cannot be provable, and, therefore neither can A be provable!

12 This is problematic. It tells us virtually any conceivable axiomatic foundation of mathematics is incomplete. If one is complete, it is inconsistent (and therefore worthless). One problem this introduces is that a true theorem may be impossible to prove; but, it turns out, we can never know that in advance of its proof if it is provable.

second incompleteness theorem

13 But it gets worse: Gödel’s second incompleteness theorem shows that such systems cannot even be shown to be consistent! This means, at any moment, someone could find an inconsistency in mathematics, and not only would we lose some of the theorems: we would lose them all. This is because, by what is called the material implication (Kline, Mathematics: The Loss of Certainty, pp. 187-8, 264), if one contradiction can be found, every proposition can be proven from it. And if this is the case, all (even proven) theorems in the system would be suspect.

material implication

Banach–Tarski paradox

14 Even though no contradiction has yet appeared in ZFC, its axiom of choice, which is required for the proof of most of what has thus far been proven, generates the Banach-Tarski paradox that says a sphere of diameter x can be partitioned into a finite number of pieces and recombined to form two spheres of diameter x. Troubling, to say the least! Attempts were made for a while to eliminate the use of the axiom of choice, but our buddy Gödel later proved that if ZF is consistent, so is ZFC (ibidem, p. 267).

Mathematics is considered empirical

15 Since its inception, mathematics has been applied extensively to the modeling of the world. Despite its cracked foundations, it has striking utility. Many recent leading minds of mathematics, philosophy, and science suggest we treat mathematics as empirical, like any science, subject to its success in describing and predicting events in the world. As Kline[18] summarizes,

empirical

> The upshot […] is that sound mathematics must be determined not by any one foundation which may some day prove to be right. The "correctness" of mathematics must be judged by its application to the physical world. Mathematics is an empirical science much as Newtonian mechanics. It is correct only to the extent that it works and when it does not, it must be modified. It is not a priori knowledge even though it was so regarded for two thousand years. It is not absolute or unchangeable.

18. Kline, Mathematics: The Loss of Certainty.